Comparing the local gravity field recovery based on radial base functions with the boundary element method

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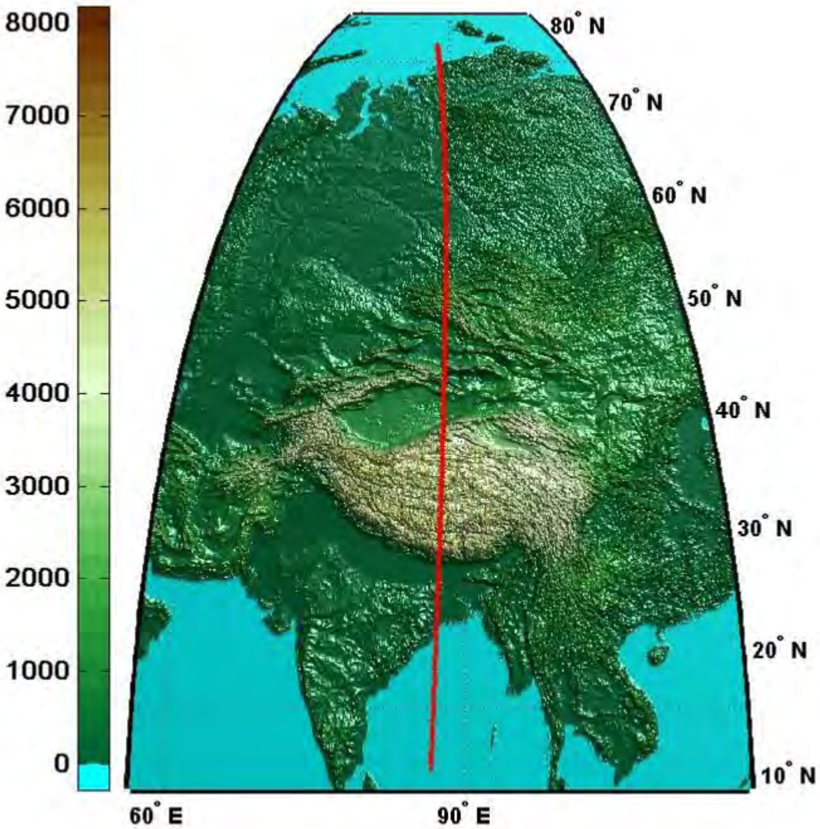
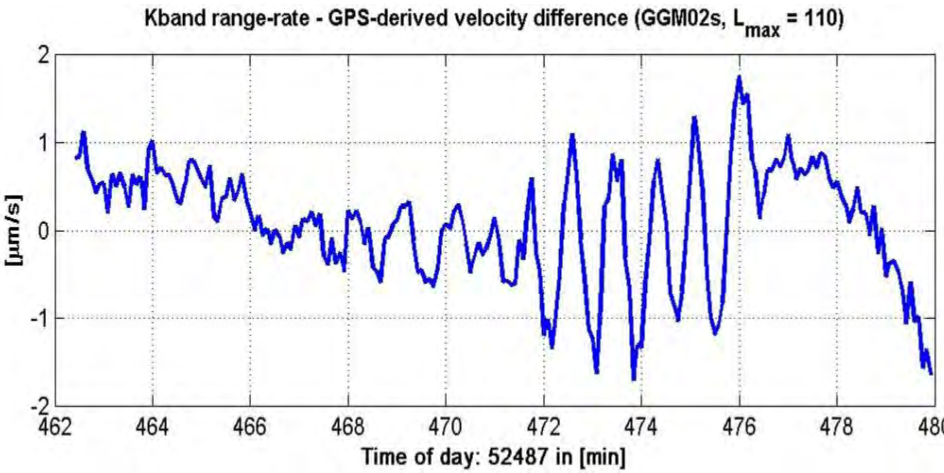
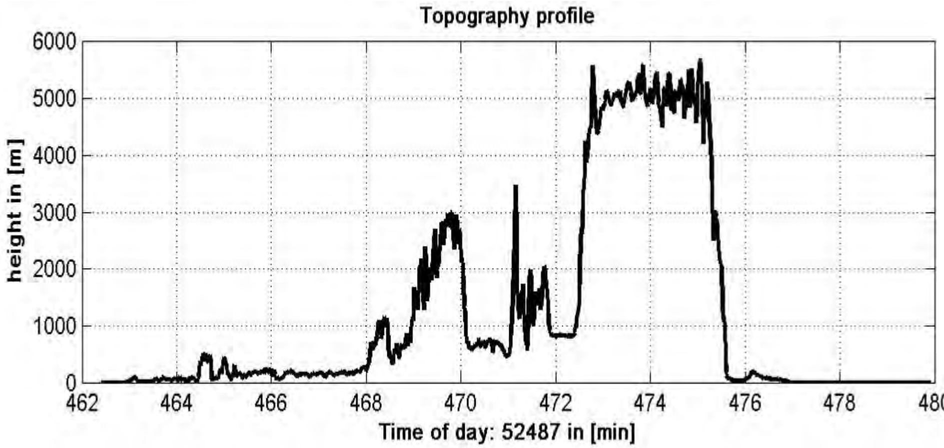
Motivation

- Residual signal







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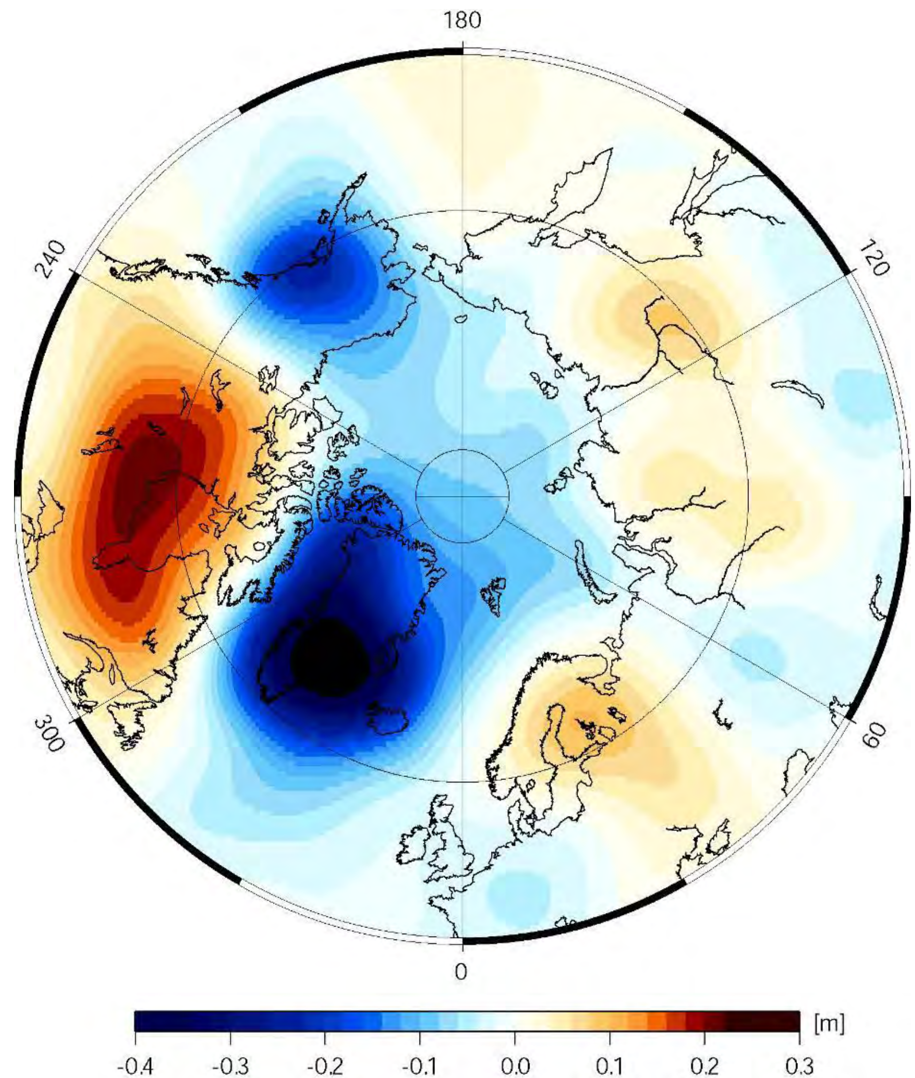
Motivation

- Leakage – problem:

  No function can be space-limited

  and band-limited at the same time

- Example:
  - Total mass change in equivalent water height
  - CSR GRACE-solutions for a six year period
  - Gauss filter with radius 500km.



Courtesy of Oli Baur, Geodetic Institute Stuttgart

Methodology

- Position-optimized Radial Base Functions
- Boundary Element Method

Methodology

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- Boundary Element Method

Position-optimized Radial Base Functions

Modelling the (residual) signal by superposition of localizing radial base functions:

$$\begin{aligned}\delta V\left(\lambda,\theta,r\right) &= \frac{GM}{R}\sum_{b=1}^{B}\eta_b\Psi\left(\sigma_b,\varpi_b,r\right)\\ &= \frac{GM}{R}\sum_{b=1}^{B}\eta_b\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+1}\sigma_b\left(n\right)P_n\left(\cos\varpi_b\right)\end{aligned}$$

with:	η_b	scale factor
	$\sigma_b\left(n\right)$	shape parameter
	ϖ_b	spherical distance to the center of the base function
	$P_n\left(\cos\varpi_b\right)$	Legendre polynomial
	λ,θ,r	spherical coordinates of the point of interest
	GM	gravitational constant
	R	Earth radius

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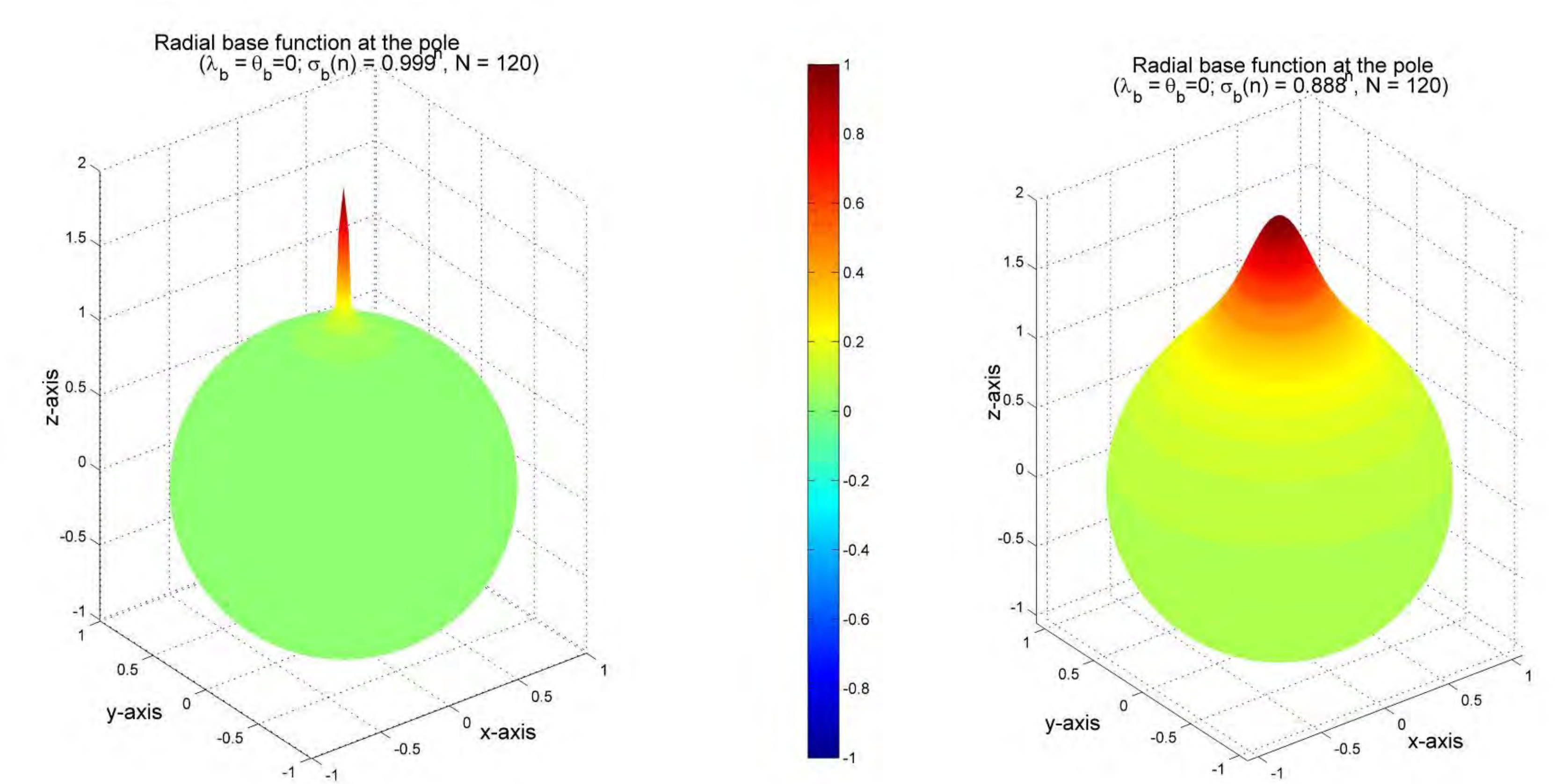
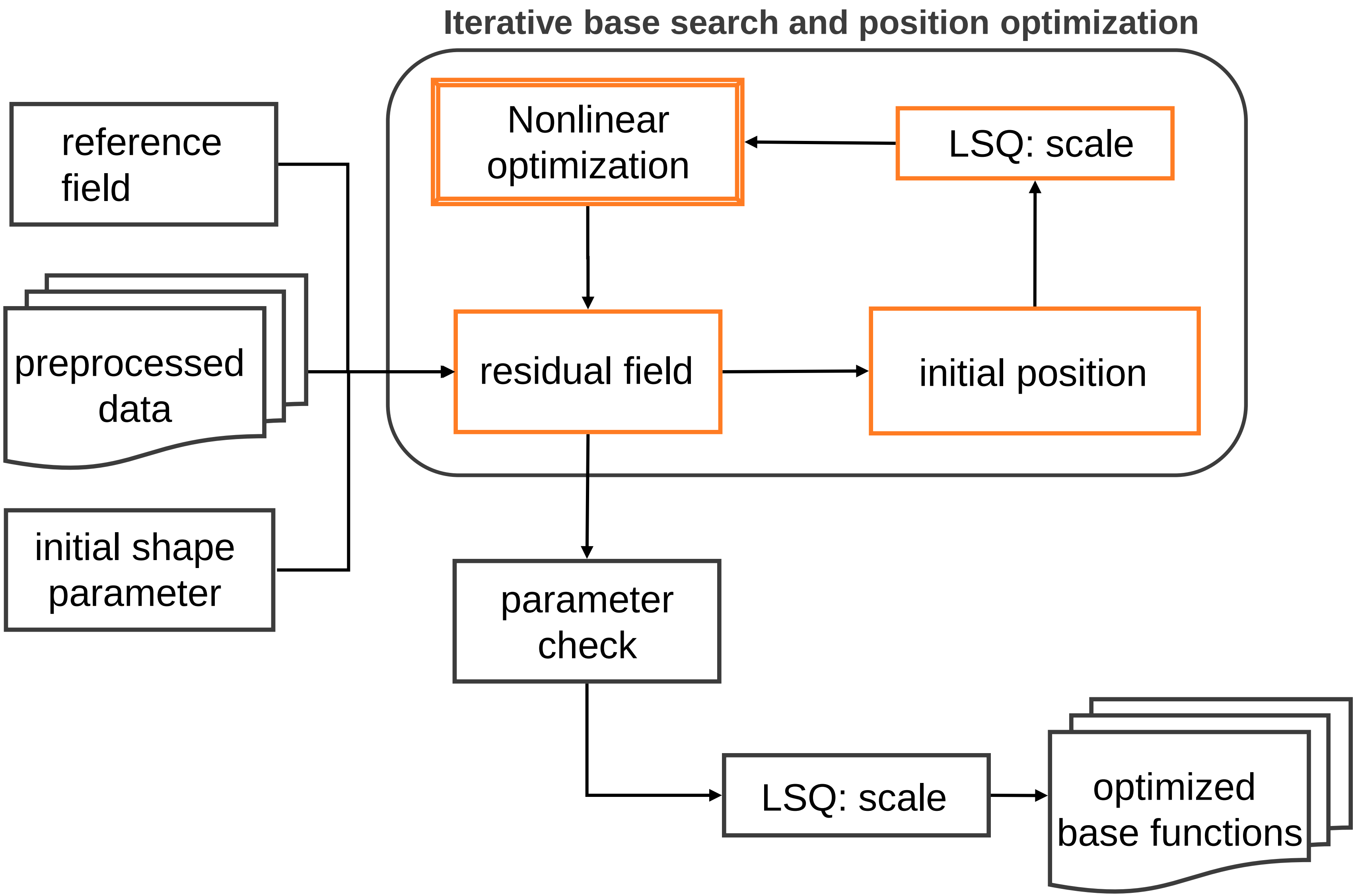
Properties

- localizing system of base functions
- isotropic = symmetric to the center point
- parameter $\sigma_b(n)$ defines shape



Position-optimized Radial Base Functions



Methodology

- Position-optimized Radial Base Functions
- Boundary Element Method

Motivation of Boundary Element Method

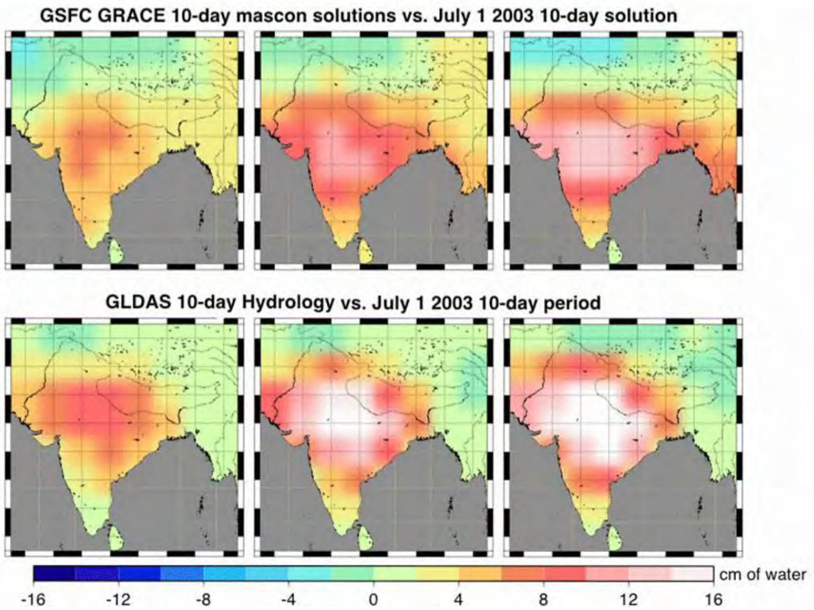
- Mascon – approach by Lemoine et al. (2007), Rowlands et al. (2007)
  - successfull modelling of GRACE monthly variations
  - use of a small additional layer

$$\Delta A_{lm}(t) = \frac{(1+k_1')\,R^2\sigma(t)}{M(2l+1)} \int_\Omega Y_{lm}\,d\Omega$$

  - use of partial derivatives w.r.t. SH-coefficients:

$$\frac{\partial \mathbf{x}}{\partial \sigma_i} = \sum_{lm} \frac{\partial \mathbf{x}}{\partial \Delta C_{lm}} \frac{\partial \Delta C_{lm}}{\partial \sigma_i} + \frac{\partial \mathbf{x}}{\partial \Delta S_{lm}} \frac{\partial \Delta S_{lm}}{\partial \sigma_i}$$

- Possible improvements:
  - use $\frac{\partial \mathbf{x}}{\partial \sigma_i}$ directly
  - use elements with a finite support
- Here: test the approximation quality of different shapes



Courtesy of Lemoine et al. (2007), http://grace.sgt-inc.com/index.html

Boundary Element Method

- Modelling the potential of a single layer

$$V(\mathbf{x}) = \int\limits_{\Omega} \frac{\sigma(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} d\Omega$$

- Decomposing the boundary into finite elements:

$$\Omega = \bigcup_{i=1}^{N} \Omega_i$$

- Assuming a constant behavior of surface mass densities within an element

$$\sigma\big|_{\Omega_i} = \sigma_i = \text{const.}$$

$$V(\mathbf{x}) = \sum_{i=1}^{N} \sigma_i \int\limits_{\Omega_i} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} d\Omega_i$$

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Boundary Element Method - Rectangles

- Considering regular rectangles:

$$\Omega_i = \{(\lambda, \phi)\,|\lambda_i \leq \lambda \leq \lambda_i + \Delta\lambda_i, \phi_i \leq \phi \leq \phi_i + \Delta\phi_i\}$$

- Potential:

$$\begin{aligned} V(\mathbf{x}) &= \sum_{i=1}^{N} \sigma_i \int\limits_{\Omega_i} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} d\Omega_i \\ &= \sum_{i=1}^{N} \sigma_i \int\limits_{\lambda_i}^{\lambda_i+\Delta\lambda_i} \int\limits_{\phi_i}^{\phi_i+\Delta\phi_i} \frac{R^2 \cos\phi \, d\phi_i d\lambda_i}{\|\mathbf{x} - (R\cos\phi\cos\lambda, R\cos\phi\sin\lambda, R\cos\phi)^T\,\|} \end{aligned}$$

- Discontinous and non-differentiable elements
- Numerical quadrature
- Many (small) elements for smooth surfaces ⇨ Regularization

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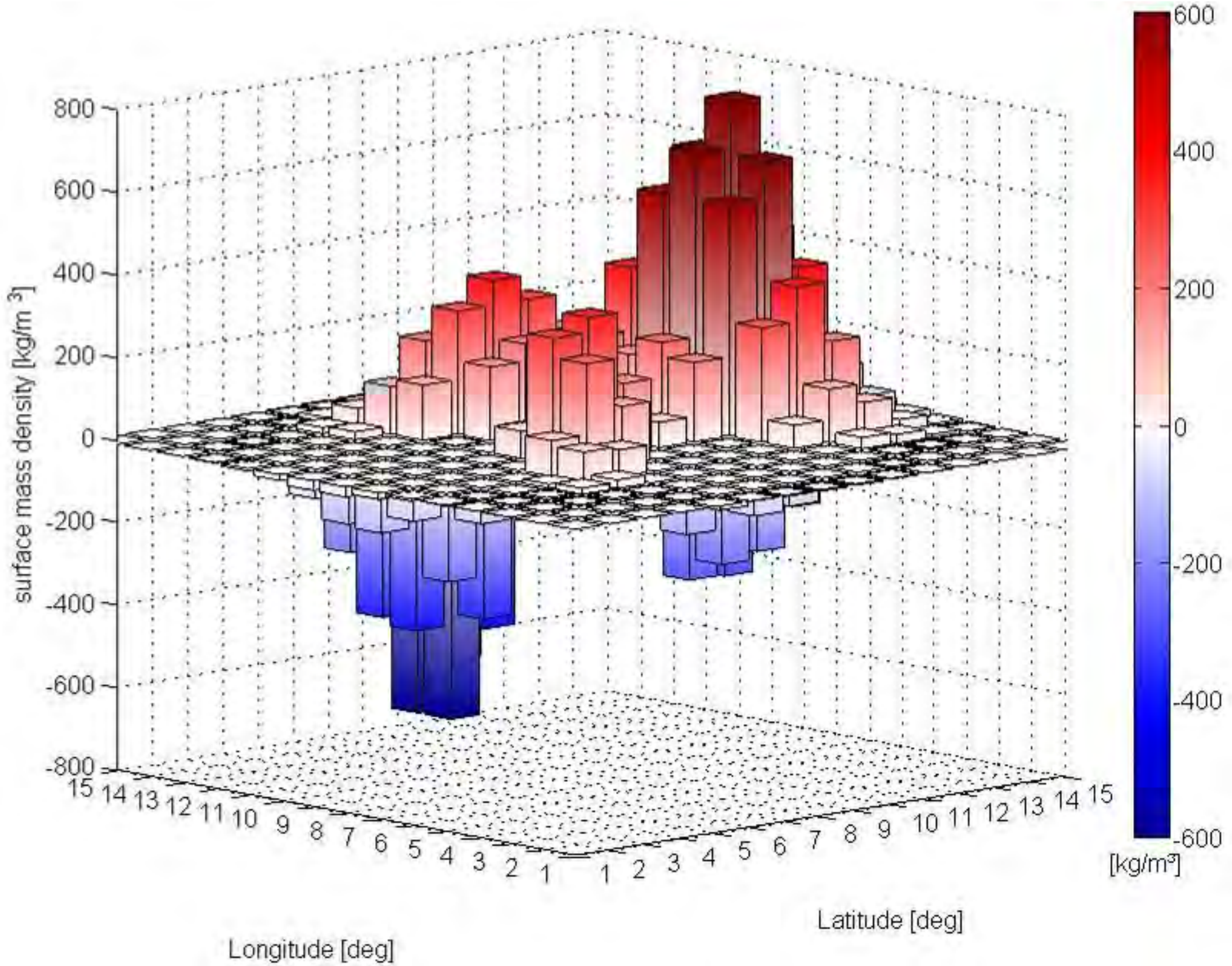
Boundary Element Method - Rectangles

- Example for rectangles



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Boundary Element Method - Triangles

- Considering triangles and linear interpolation of the surface mass densities **and** the kernel within a triangle

$$\kappa_i(\mathbf{x},\lambda,\phi) = \frac{\sigma_{i,1}}{\|\mathbf{x}-\mathbf{y}(\phi_{i,1},\lambda_{i,1})\|}\Phi_{i,1} + \frac{\sigma_{i,2}}{\|\mathbf{x}-\mathbf{y}(\phi_{i,2},\lambda_{i,2})\|}\Phi_{i,2} + \frac{\sigma_{i,3}}{\|\mathbf{x}-\mathbf{y}(\phi_{i,3},\lambda_{i,3})\|}\Phi_{i,3}$$

- Potential:

$$\begin{aligned} V(\mathbf{x}) &= \sum_{i=1}^{N} \int_{\Omega_i} \frac{\sigma_i}{\|\mathbf{x}-\mathbf{y}_i\|} d\Omega_i \\ &= \sum_{i=1}^{N}\sum_{k=1}^{3} \frac{\sigma_k}{\|\mathbf{x}-\mathbf{y}(\lambda_k,\phi_k)\|} \int_0^1 \int_0^{1-\xi} \Phi_{ik}|J|\,d\eta\,d\xi \end{aligned}$$

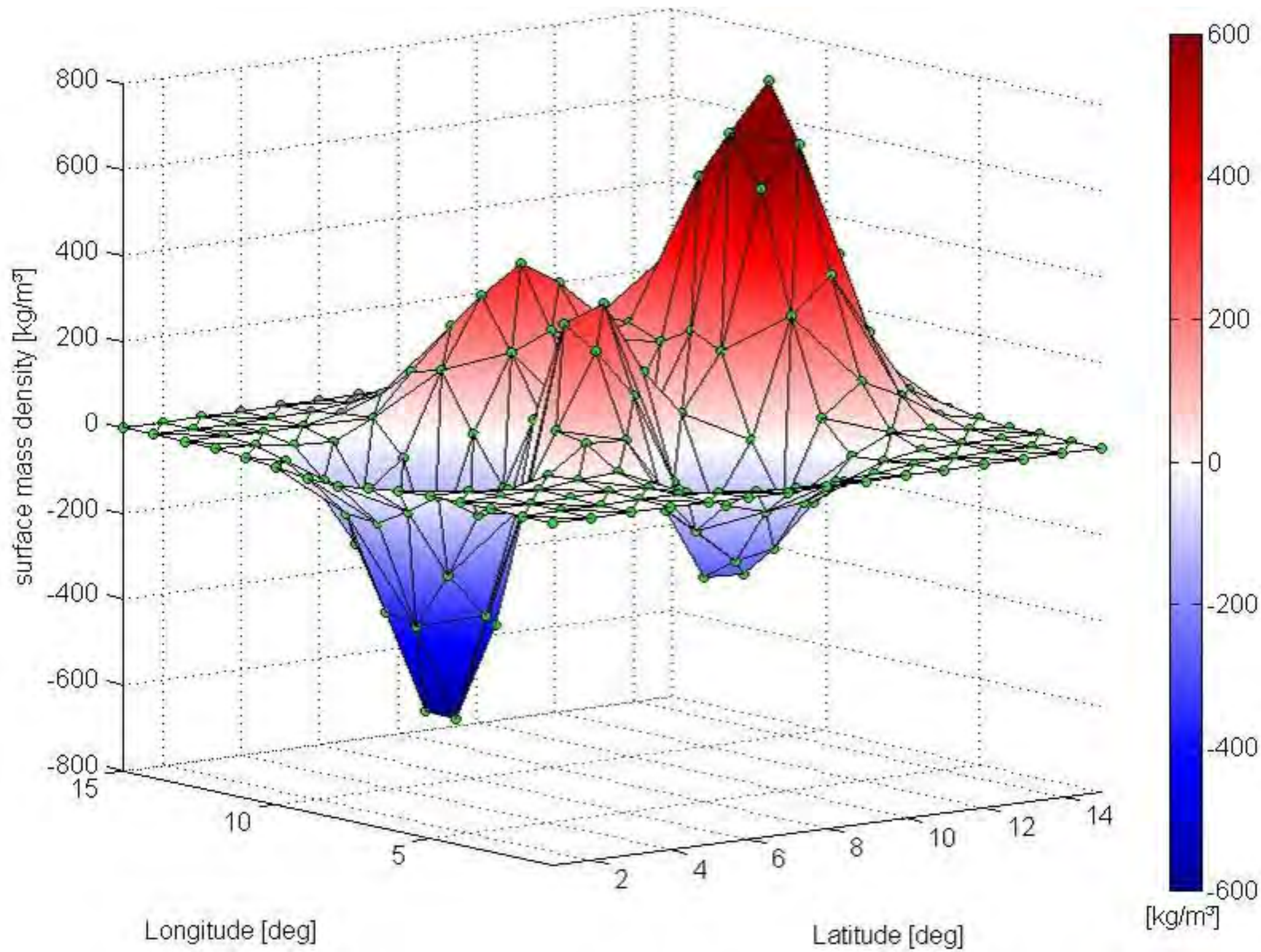
- with

$$\begin{aligned} \Phi_{i,1}(\lambda(\xi,\eta),\phi(\xi,\eta)) &= 1-\xi-\eta \\ \Phi_{i,2}(\lambda(\xi,\eta),\phi(\xi,\eta)) &= \xi \\ \Phi_{i,3}(\lambda(\xi,\eta),\phi(\xi,\eta)) &= \eta \end{aligned}$$

- Continuous but non-differentiable elements
- Analytical solution of the normal triangle

Boundary Element Method - Triangles

- Example for triangles



Simulation study

a) Single point mass

b) Multiple point masses forming a residual field

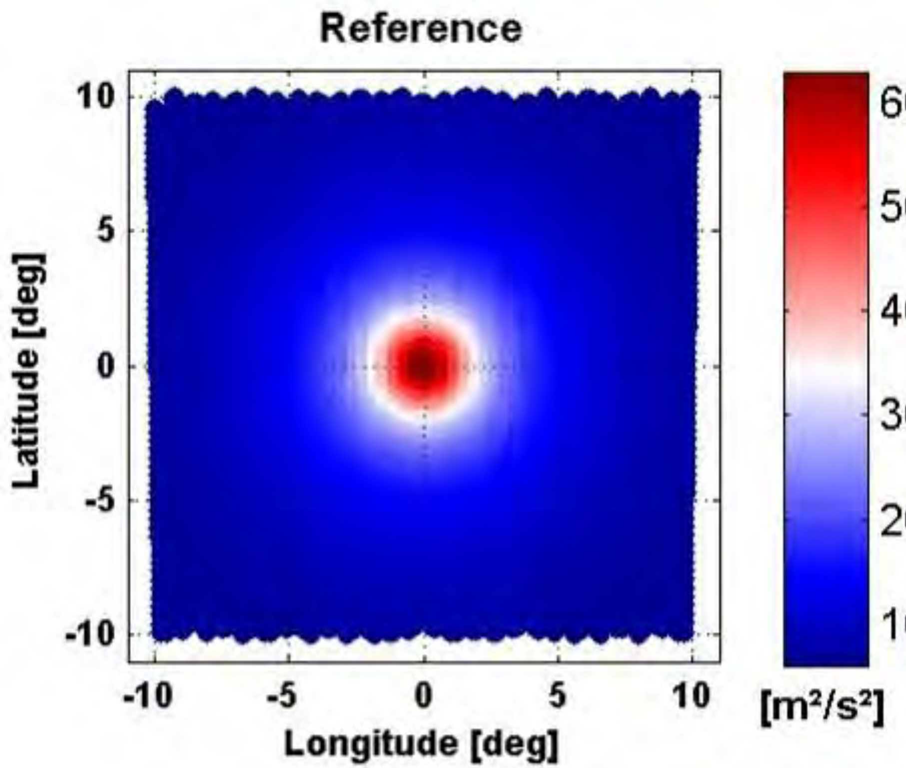
Simulation study

a) Single point mass

b) Multiple point masses forming a residual field

a) Single point mass

- Single point mass at depth 125km
- Area: 20° x 20°
- Keplerian orbit
  - height = 385 km
  - 30 days
  - 5 second sampling
  - 3204 observation



- Pseudo-observation: potential energy

$$V(\lambda, \phi, r) = \frac{2 \cdot 10^{-8} \cdot GM}{\sqrt{(R-d)^2 + r^2 - 2\,r\,(R-d)\cos\psi}}$$

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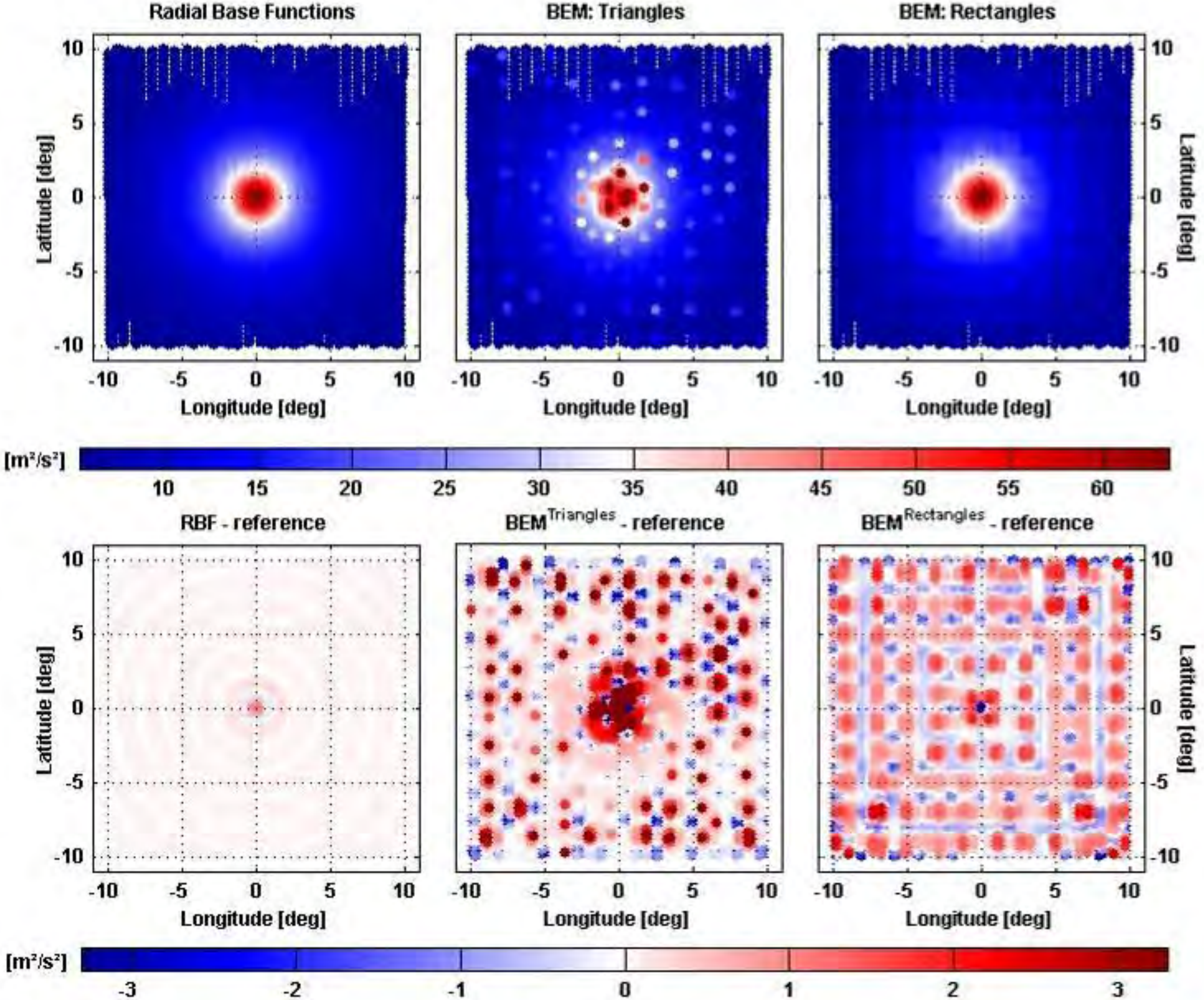
a) Single point mass – BEM at depth 10km



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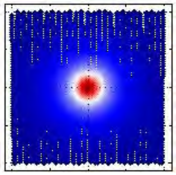
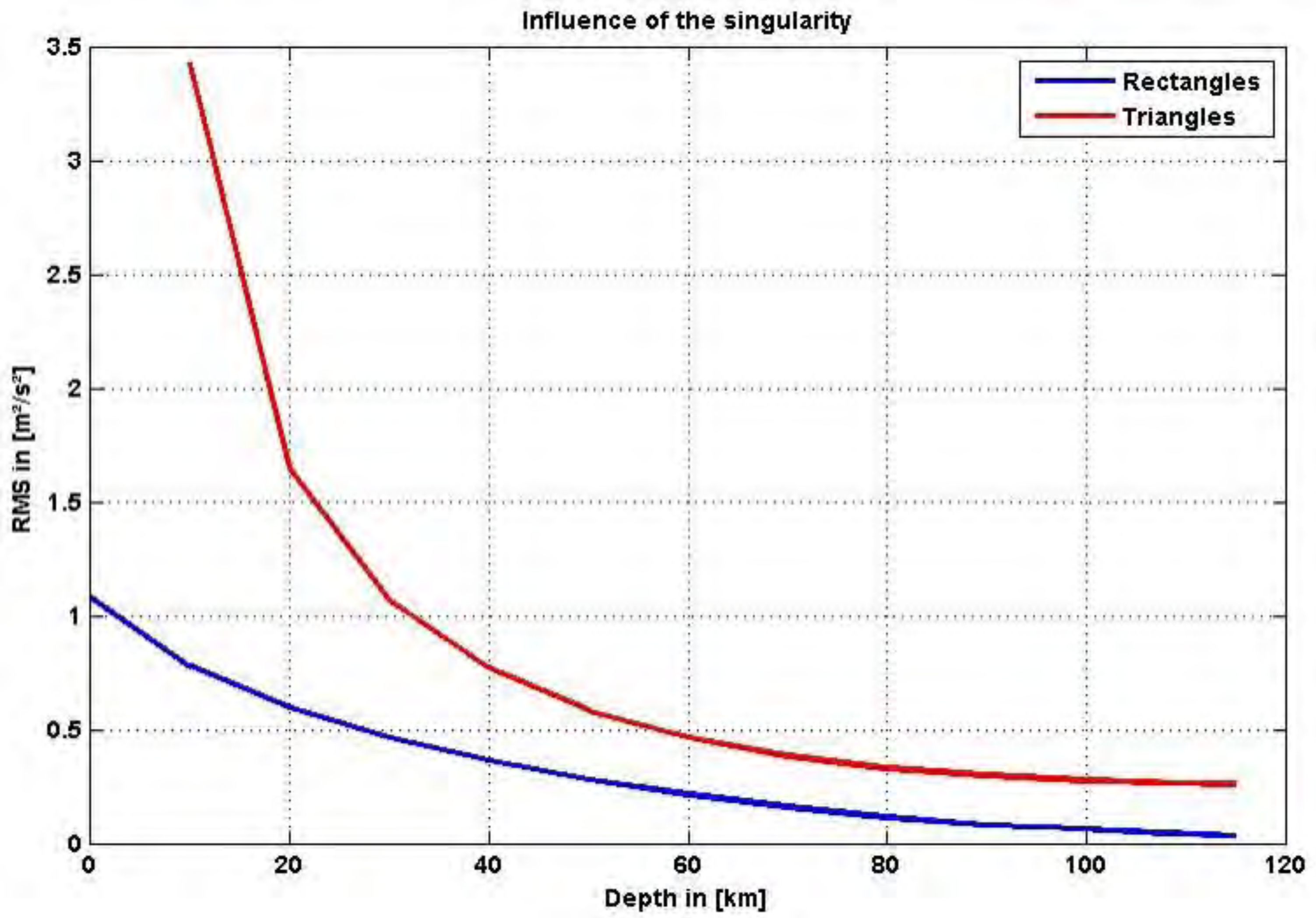
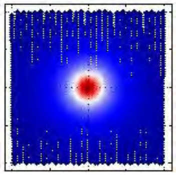
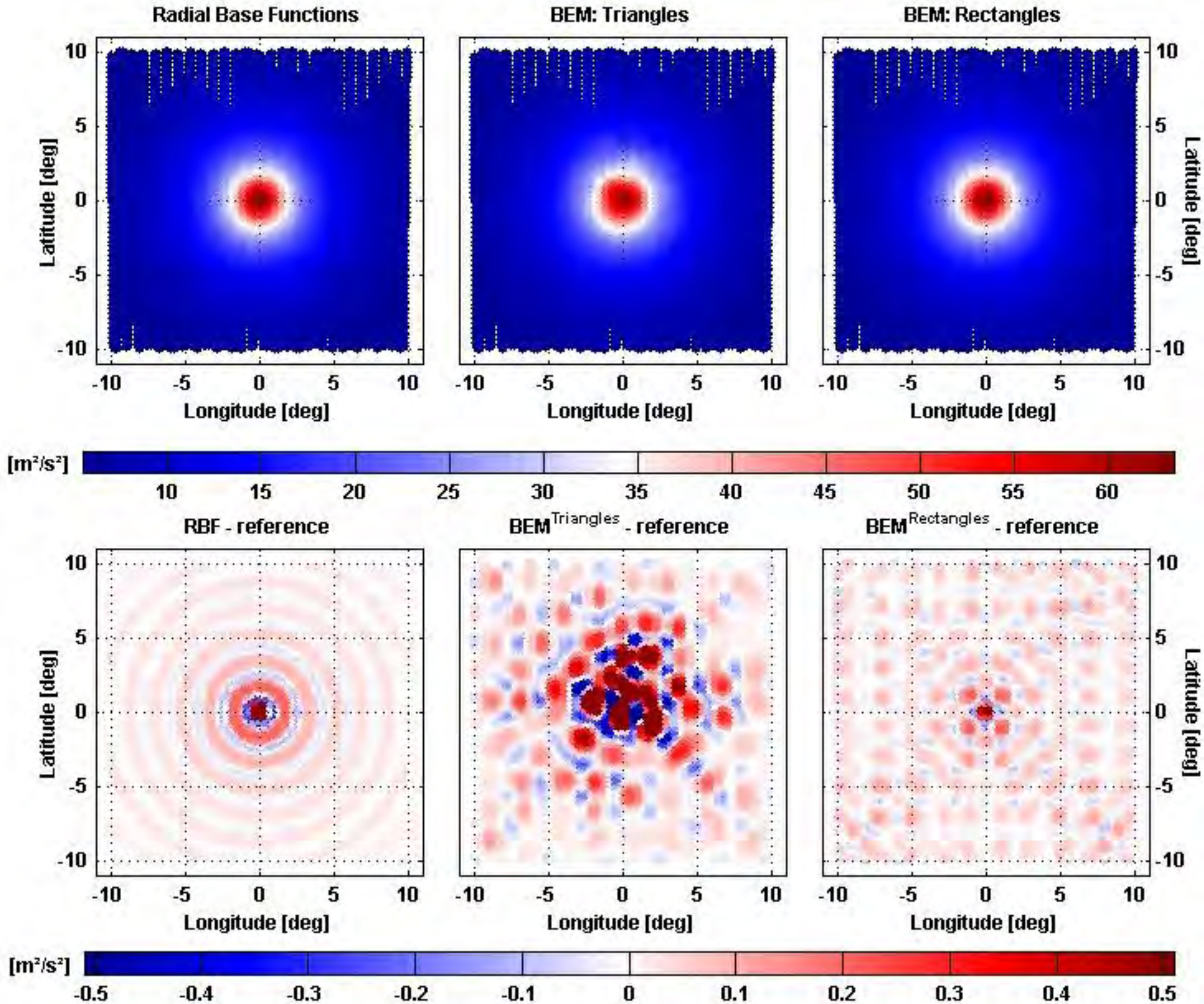
a) Single point mass



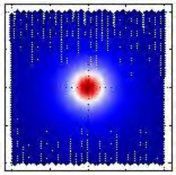


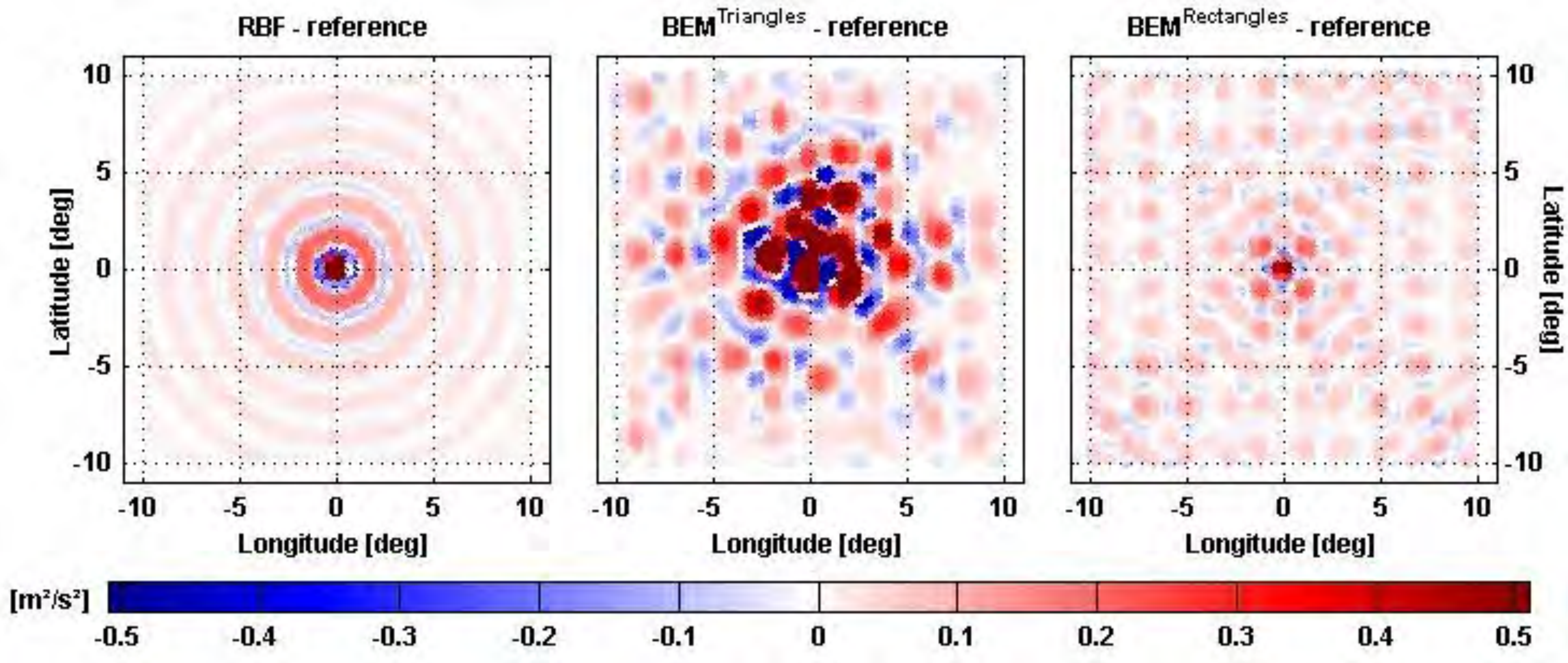
a) Single point mass – BEM at depth 110km





a) Single point mass - BEM at depth 110km





Statistics:

	RMS		Max		Min		Corr.
	[m²/s²]	Rel. %	[m²/s²]	Rel. %	[m²/s²]	relativ %	%
RBF	0.055	0.40	0.382	0.61	-1.085	20.99	99.0
BEM (Triangle)	0.266	1.94	3.157	4.99	-3.564	68.98	98.2
BEM (Rectangle)	0.043	0.31	0.226	0.36	-0.501	9.70	99.6

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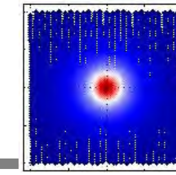
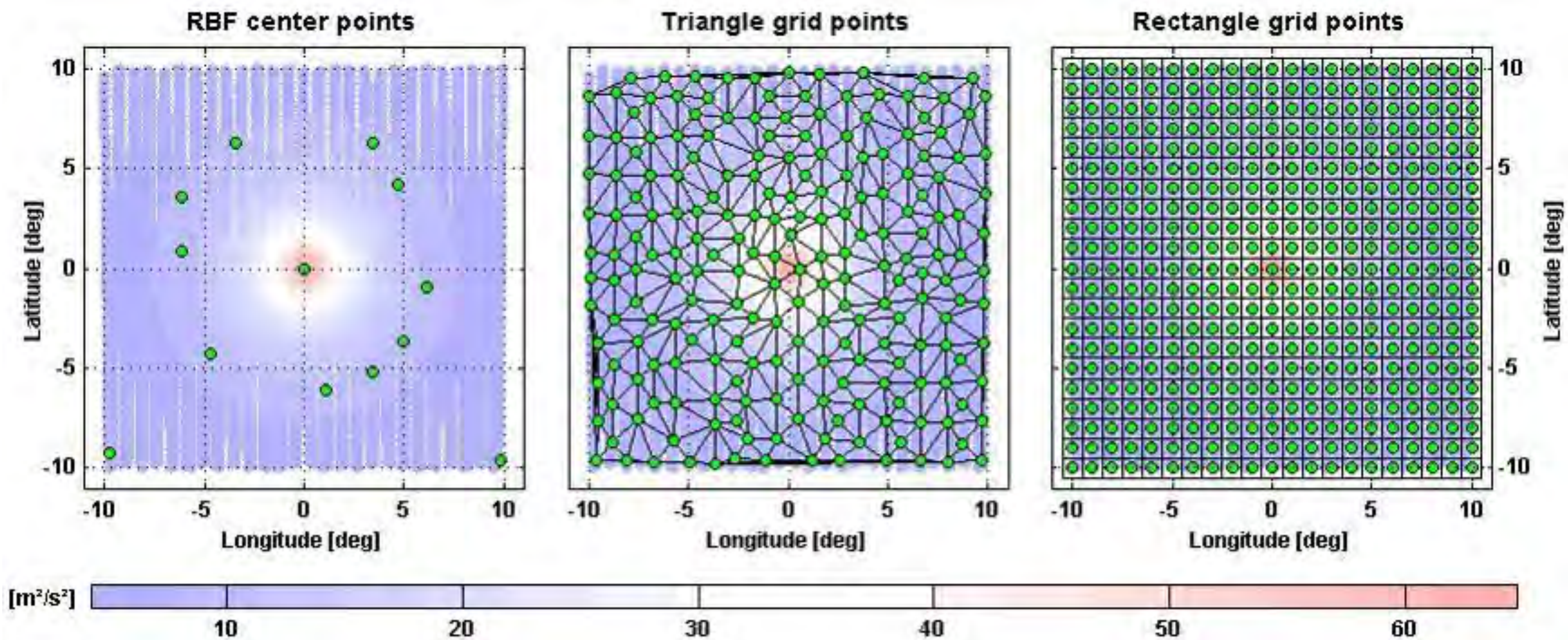
a) Single point mass - BEM at depth 110km





	Number of elements	Regularization
RBF	13	No
BEM (Triangle)	209	No
BEM (Rectangle)	441	Yes (409)

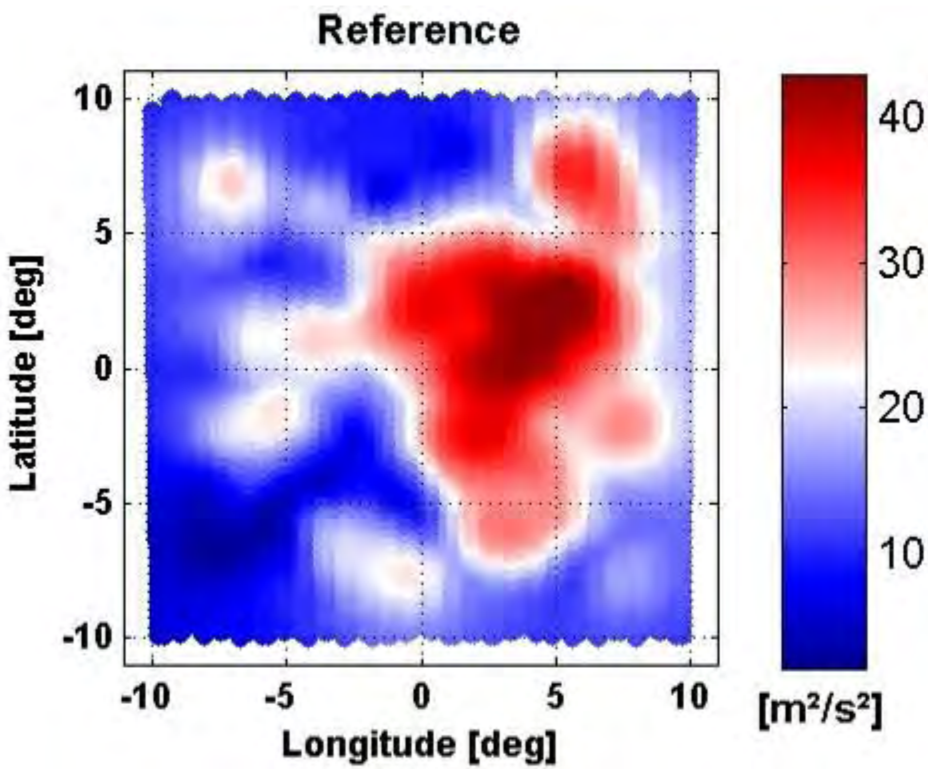
Simulation study

a) Single point mass

b) Multiple point masses forming a residual field

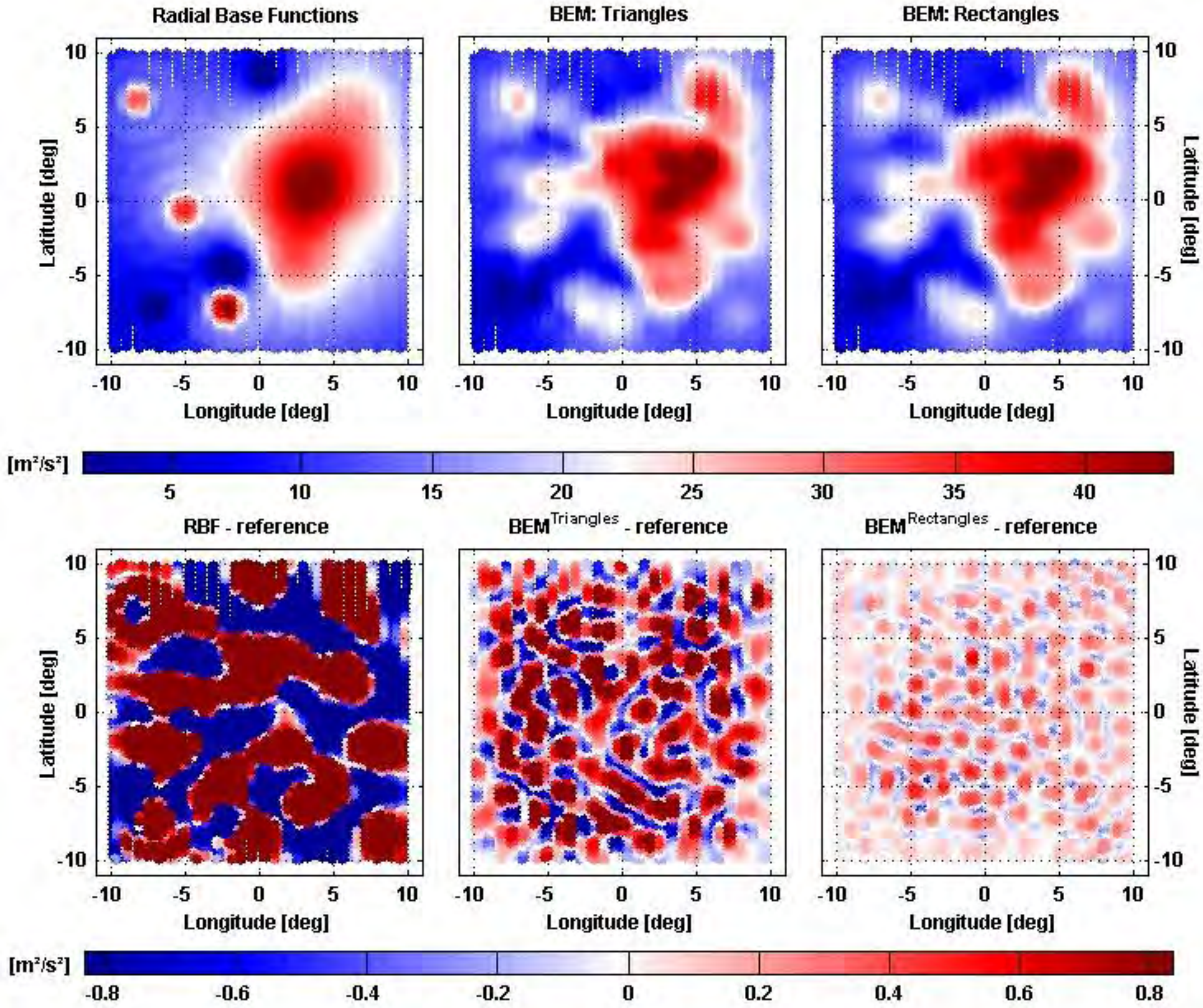
Simulated residual field

- 4225 point masses at depth 120km – 130km
- Area: 20° x 20°
- Keplerian orbit
  - height = 385 km
  - 30 days
  - 5 second sampling
  - 3204 observation
- Pseudo-observation: potential energy

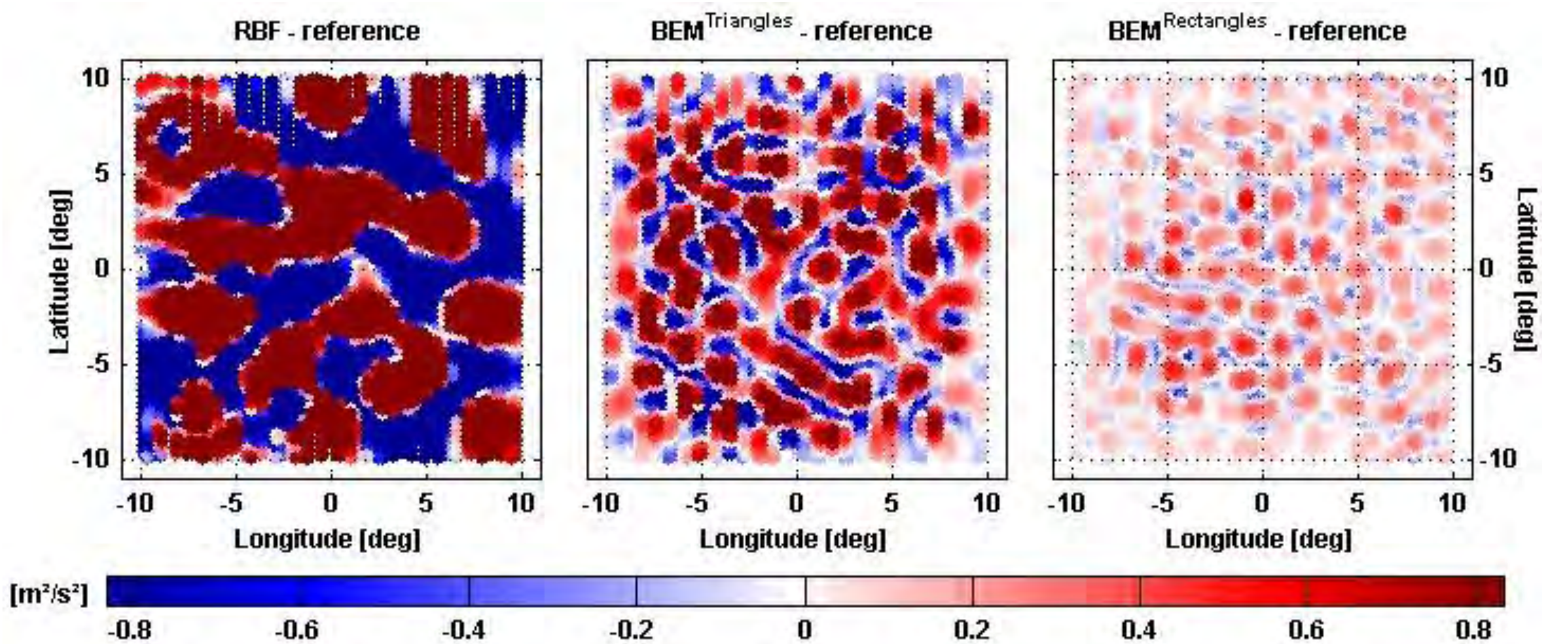


$$V(\lambda, \phi, r) = \sum_{i=1}^{4225} \frac{\sigma_i \cdot GM}{\sqrt{(R - d_i)^2 + r^2 - 2\,r\,(R - d_i)\cos\psi_i}}$$

Simulated residual field - BEM at depth 110km

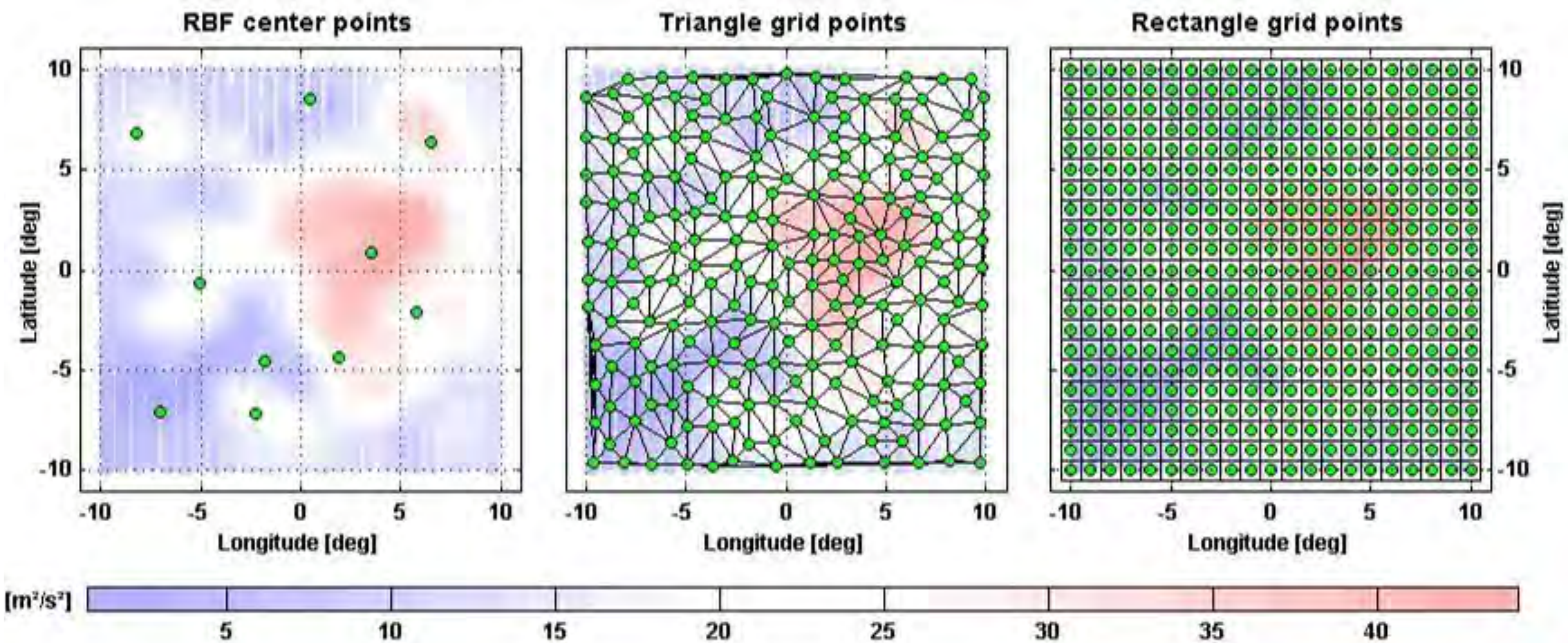


Simulated residual field - BEM at depth 110km



Statistics:	RMS		Max		Min		Corr.
	[m²/s²]	Rel. %	[m²/s²]	Rel. %	[m²/s²]	Rel. %	%
RBF	3.734	18.00	24.047	54.67	-17.341	996.60	58.3
BEM (Triangle)	0.675	3.25	2.703	6.15	-3.368	193.58	92.2
BEM (Rectangle)	0.153	0.74	0.821	1.87	-0.570	32.74	98.3

Simulated residual field - BEM at depth 110km



	Number of elements	Regularization
RBF	10	No
BEM (Triangle)	209	No
BEM (Rectangle)	441	Yes (416)

Conclusions

Conclusions

- Position-optimized radial base functions for distinct features
  - number of parameter is small (4 x number of bases)
  - problem is non-linear
- Boundary element method for smooth features
  - preferably continuous/differentiable elements (no regularization)
  - grid?
  - preferably numerical quadrature of the Kernel

Outlook:

- Integration: near-zone and far-zone
  - singular, quasi-singular, regular
- Shape elements: higher order triangles and quadrilaterals
- Partial derivatives of the range rate w.r.t. to the surface mass densities

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