Construction of directional wavelets on the sphere

Markus Antoni, Matthias Roth, Wolfgang Keller

Geodätische Woche 2014

GIS

Motivation

model refinement by localizing base functions

- tend to zero outside the area of influence
- model mainly data within the area of interest

Fig: Mascons (Lemoine, 2007)

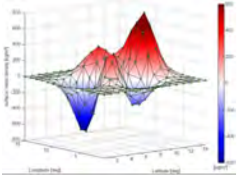

Fig: Boundary elements (Weigelt, 2012)

Motivation

model refinement by localizing base functions

- tend to zero outside the area of influence
- model mainly data within the area of interest
- very often: radial symmetric

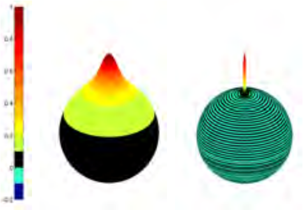

Fig: Radial base functions on a sphere

Fig: Mascons (Lemoine, 2007)

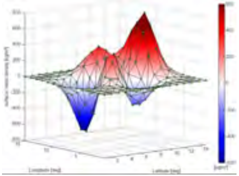

Fig: Boundary elements (Weigelt, 2012)

Motivation

observations by satellites have

- preferred direction
- converging of tracks

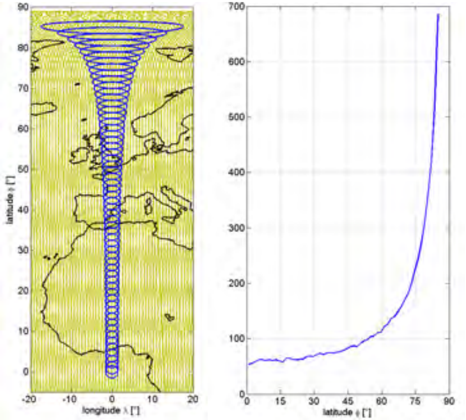

Fig: Spherical caps with radius 1.5° and points per cap

Idea

- isotropic functions $\Psi(\mathbf{x}, \mathbf{y})$ in spatial domain ($\mathbf{x}$: location, $\mathbf{y}$: node/center)
- linear transformation $\tilde{\mathbf{x}} = \underline{\mathbf{E}} \cdot \mathbf{x}$ and $\tilde{\mathbf{y}} = \underline{\mathbf{E}} \cdot \mathbf{y}$
- 'elliptical' contour lines per wavelets

Poisson wavelets

Poisson wavelets of order N:

$$\mathcal{X}_n = \left(\|\mathbf{y}\| \frac{\partial}{\partial \|\mathbf{y}\|} \right)^n \frac{1}{\|\mathbf{x}-\mathbf{y}\|}$$

for $n = 0, 1, \ldots, N+1$ and

$$\Psi(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi R^2} \left(2\mathcal{X}_{N+1} + \mathcal{X}_N\right)$$

recursive formulas up to $N = 9$

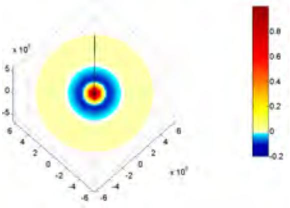

Fig: (Normalized) wavelet on the sphere

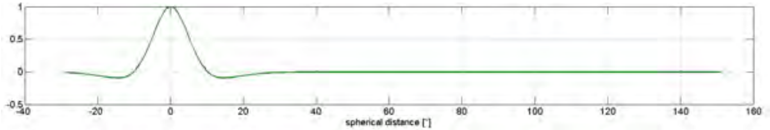

Fig: Cut along the meridian

Transformation

- keep size in North-South direction
- scaling in East-West direction
- empirical factor: $f(\phi) := \exp\left(\frac{1}{2} - \frac{1}{2}\left(\frac{\phi}{45}\right)^2\right)$

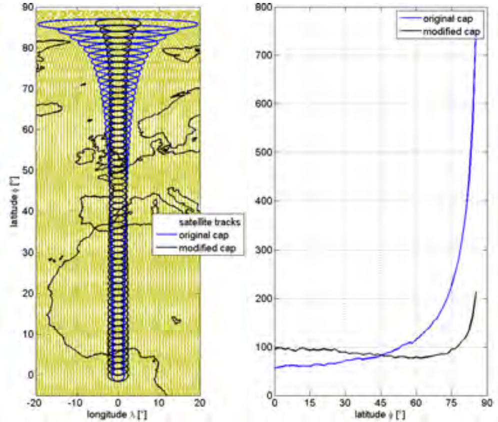

Fig: Points within modified spherical caps and points per cap

'Elliptical' wavelets

$$\underline{\mathbf{E}}^{-1} = (\underline{\mathbf{R}}_g^e)^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{\mathbf{R}}_g^e \qquad \text{with} \quad \underline{\mathbf{R}}_g^e = \underline{\mathbf{R}}_2(90-\phi)\underline{\mathbf{R}}_3(\lambda)$$

$$\Rightarrow \Psi(\underline{\mathbf{E}}\mathbf{x}, \underline{\mathbf{E}}\mathbf{y})$$

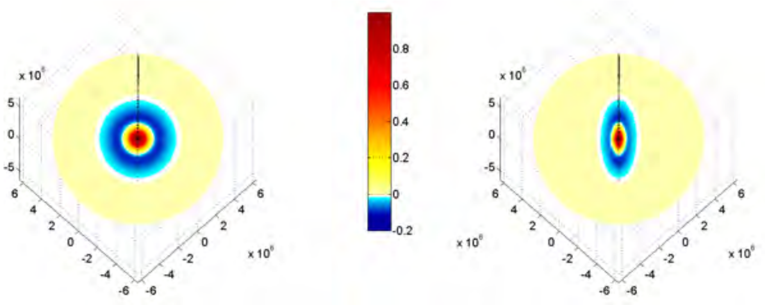

Fig: (Normalized) original and modified wavelet on the sphere

Spherical grid

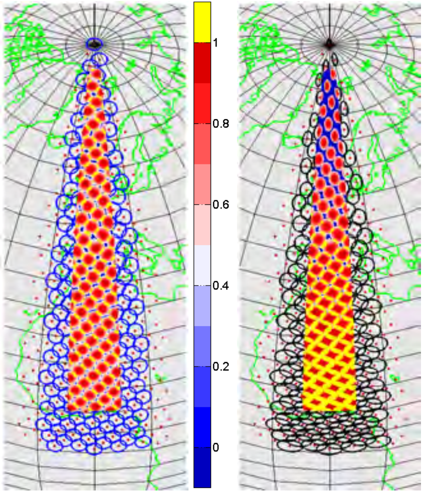

Fig: Fibonacci grid (depth = 100 km)

- well suited for standard wavelets
- not enough nodes for ‘elliptic’ wavelets

GIS

Spherical grid

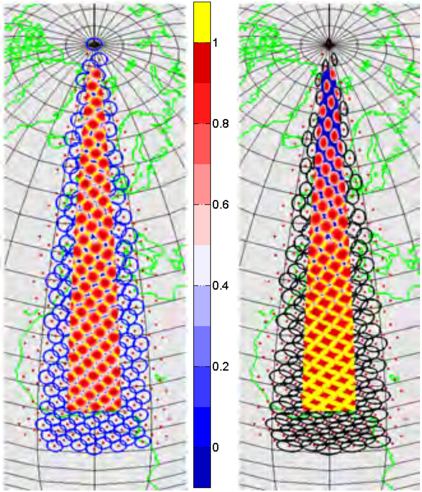

Fig: Fibonacci grid (depth = 100 km)

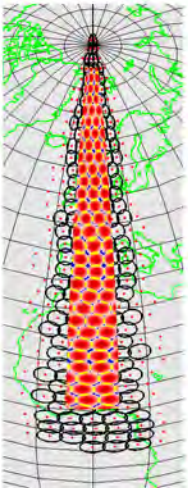

Fig: ‘scaled helix grid’

Example

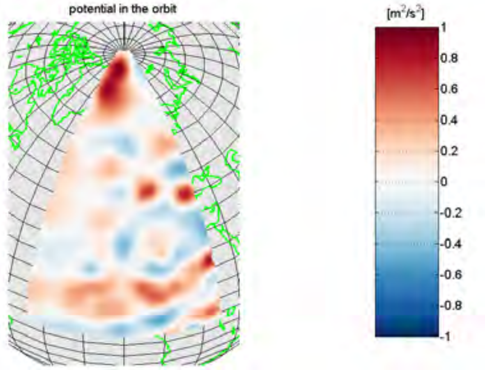

Fig: Potential in space

- GRACE-like orbit parameter
- energy balance approach
- subtraction of a reference field
- 'regional' selection
- statistic in $[\mathrm{m}^2/\mathrm{s}^2]$:

MEAN	0.1191
MAX	0.9814
MIN	-0.5464
STD	0.2774

Synthesis

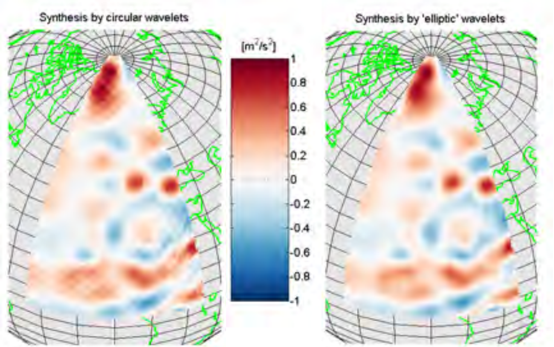

Fig: Synthesis by radial and ‘elliptic’ wavelets

	radial	‘elliptic’
nodes	623	598
cond($\underline{\mathbf{A}}^\top \underline{\mathbf{A}}$)	1709	495
regularization	$3.58 \cdot 10^{-7}$	$2.56 \cdot 10^{-7}$
correlation [%]	0.94 (0.99)	0.93 (0.99)

GIS

Residuals in the inner zone

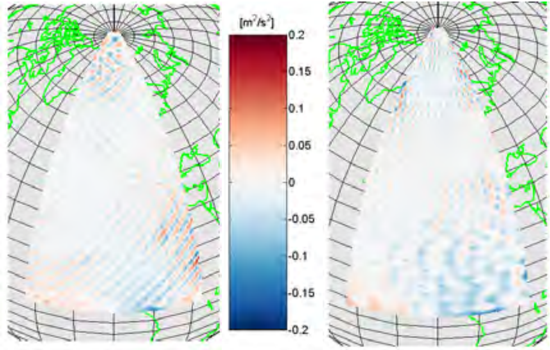

Fig: residuals after wavelet synthesis

$[\mathrm{m}^2/\mathrm{s}^2]$	synthesis		residuals	
	radial	'elliptic'	radial	'elliptic'
MAX	1.0101	1.0325	0.1623	0.2060
MIN	−0.6260	−0.6138	−0.1550	−0.1477
MEAN	0.1137	0.1095	0.0056	0.0098
STD	0.2782	0.2769	0.0283	0.0278

Comparison

So far, similar quality for both kinds of wavelets

But for ‘elliptic’ wavelets

- smaller condition number and regularization parameter
- consideration of observation geometry
- improvements by grid modifications

Summary

construction of spherical base functions

- non-isotropic/directional dependent
- re-scaling in East-West direction with latitude
- in spatial domain
- easy/fast calculation

Summary

construction of spherical base functions

- non-isotropic/directional dependent
- re-scaling in East-West direction with latitude
- in spatial domain
- easy/fast calculation

Open Questions

- scaling should depend on orbital parameters (I,r)
- unchanged caps in higher/lower latitudes?
- analysis in other directions?
- special grid design?

GIS