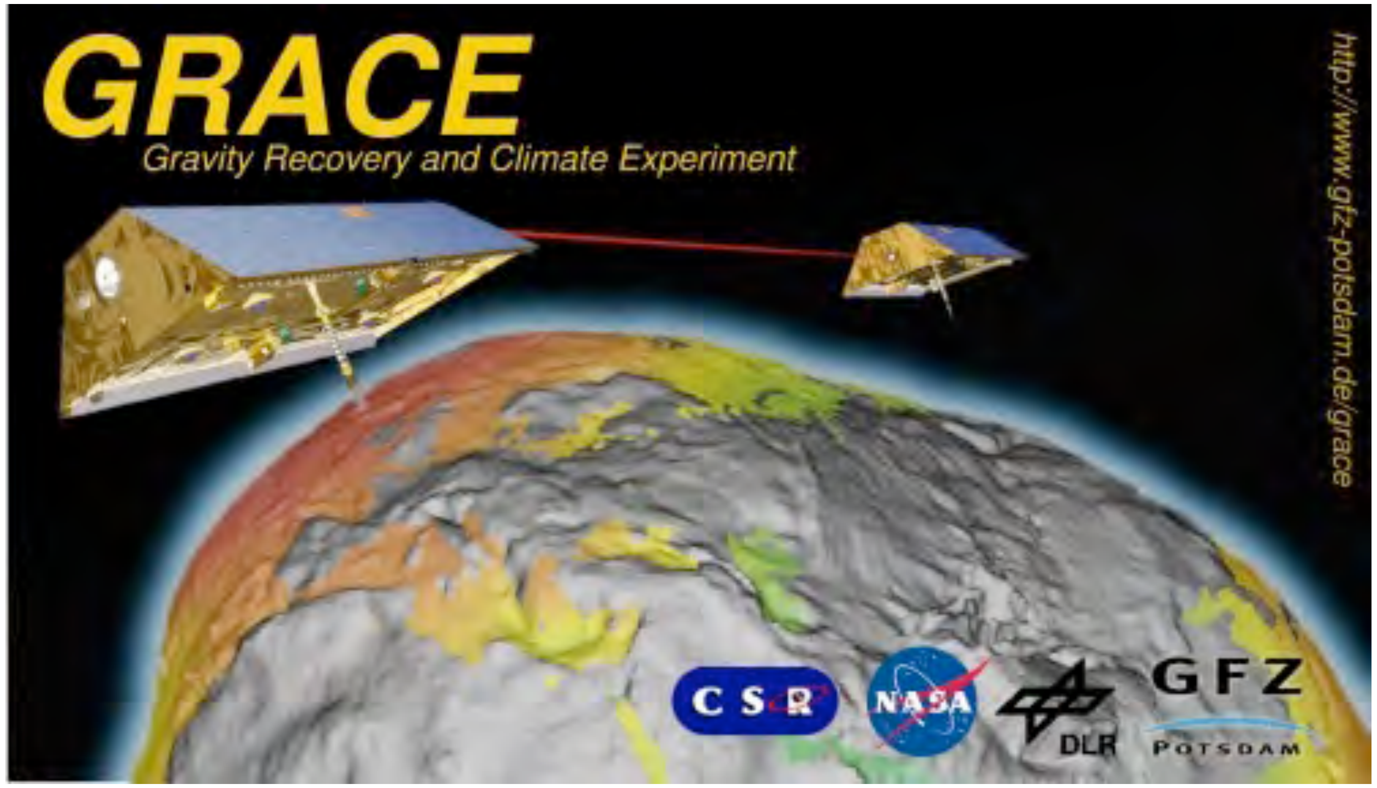
Local improvement of GRACE gravity field solutions using SO(3) representations

M. Antoni & W. Keller
Institute of Geodesy, Universität Stuttgart, Germany

7. January 2012

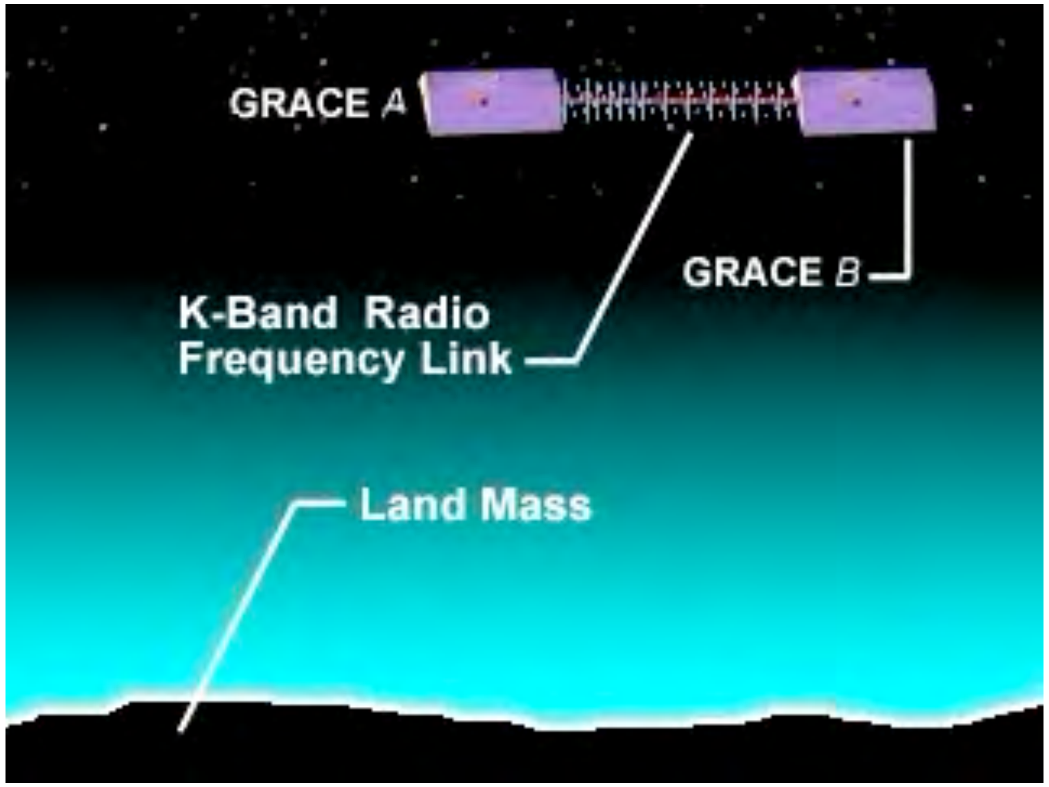
GIS

1. GRACE mission



- Two satellites in the same orbit, separated along-track by about 200km.
- Continuously, measuring their relative velocity, with very high accuracy.
- Changes in the relative velocity are related to changes in gravity:

Courtesy of CSR



1. GRACE mission (cont.)

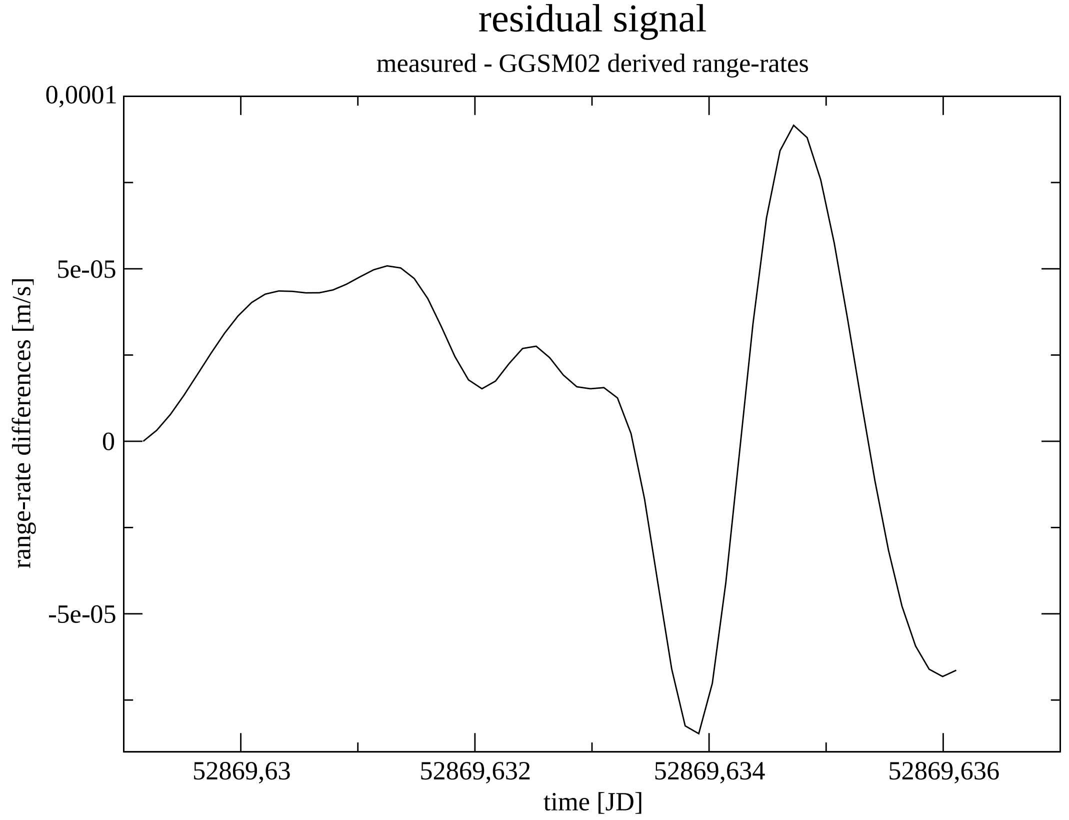
The representation of the monthly given gravity field solutions is in spherical harmonics

$$V(r,\theta,\lambda) = \frac{GM}{R}\left(1 + \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}\boxed{c_{nm}}Y_{n,m}(\theta,\lambda)\right)$$

The coefficients in this linear combination are called monthly GRACE solution and get an acronym as for instance GGM02.

GIS

2. Motivation



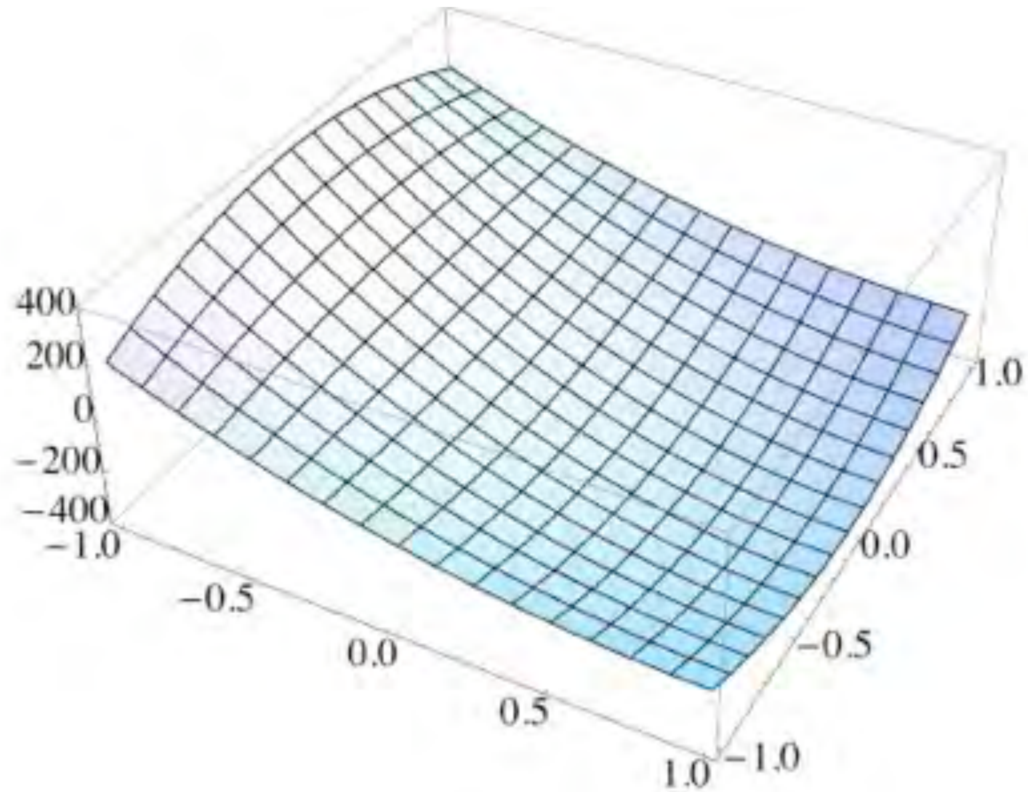
- Observed GRACE range-rates are compared to synthetic range-rates, computed from global GRACE gravity field solutions, as e.g. GGSM02

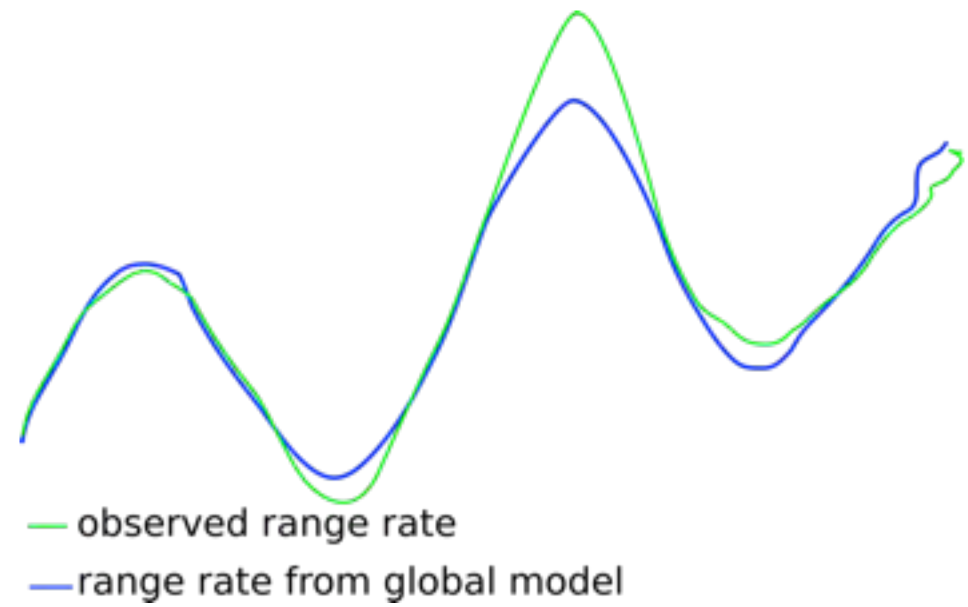
- Remaining difference is not white noise, but contains (at least partially) unresolved gravity information.

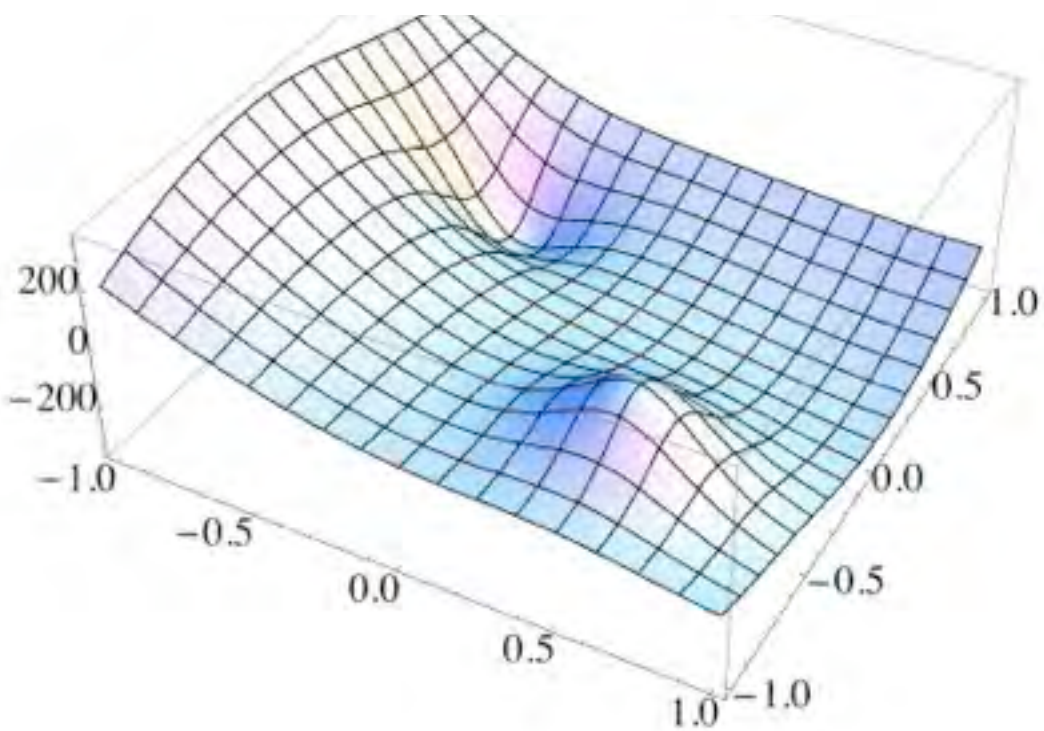
- The incomplete exploitation of range-rate information is due to the global support of the base functions

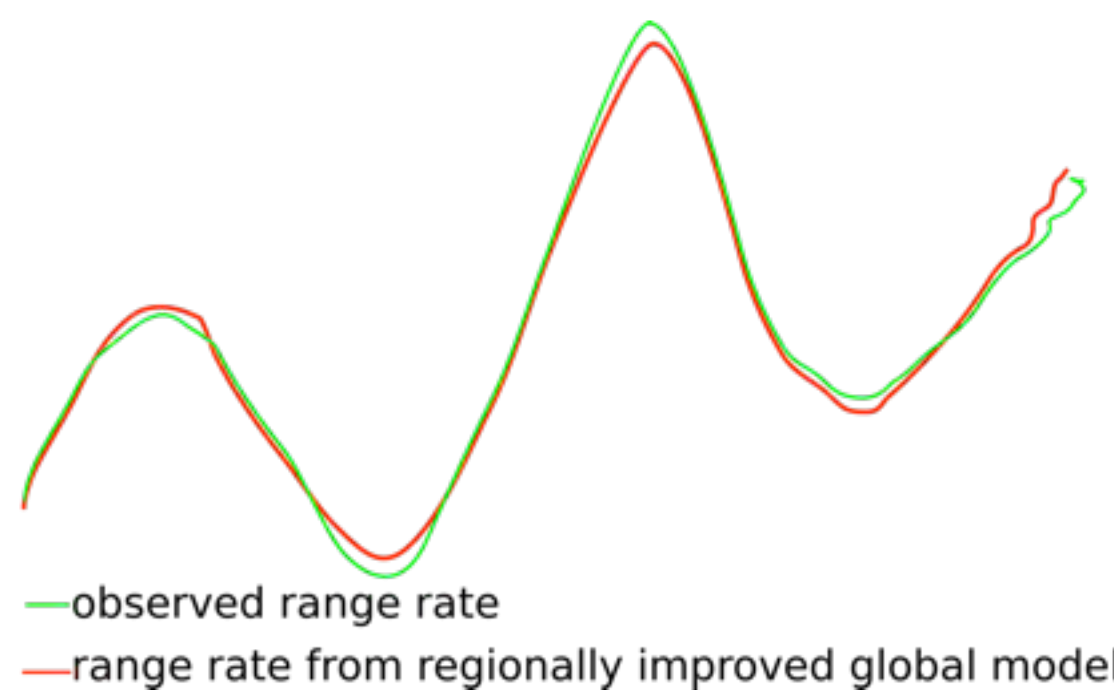
→ Unresolved gravity information can be resolved regionally by modeling it by radial base functions

1. Motivation (cont.)



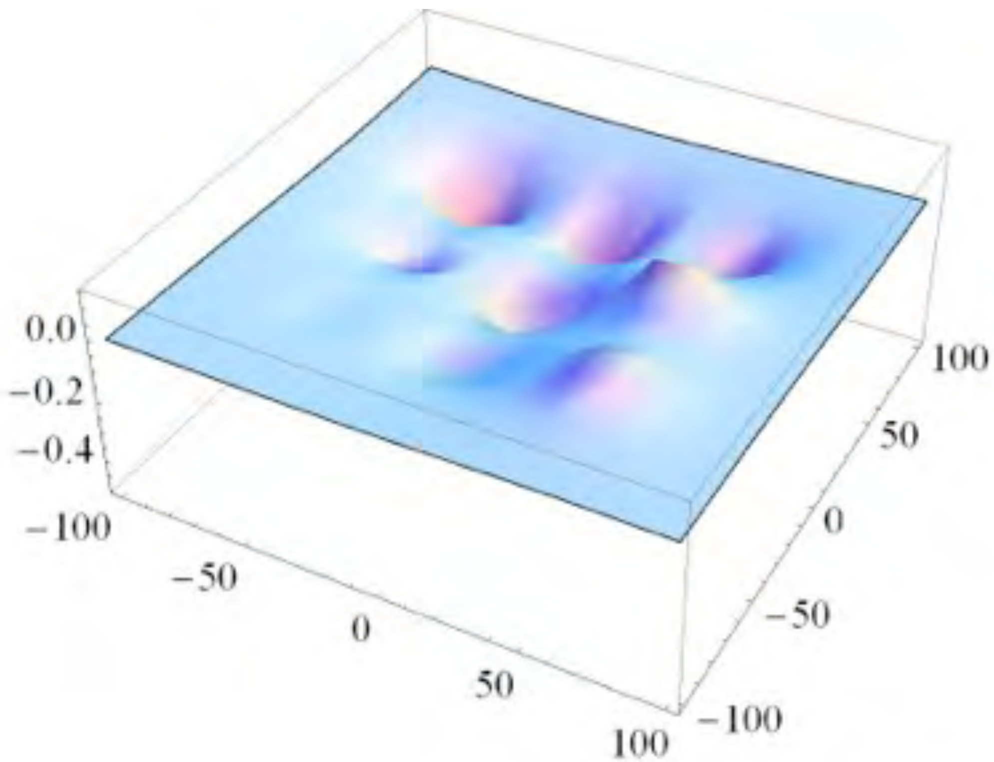


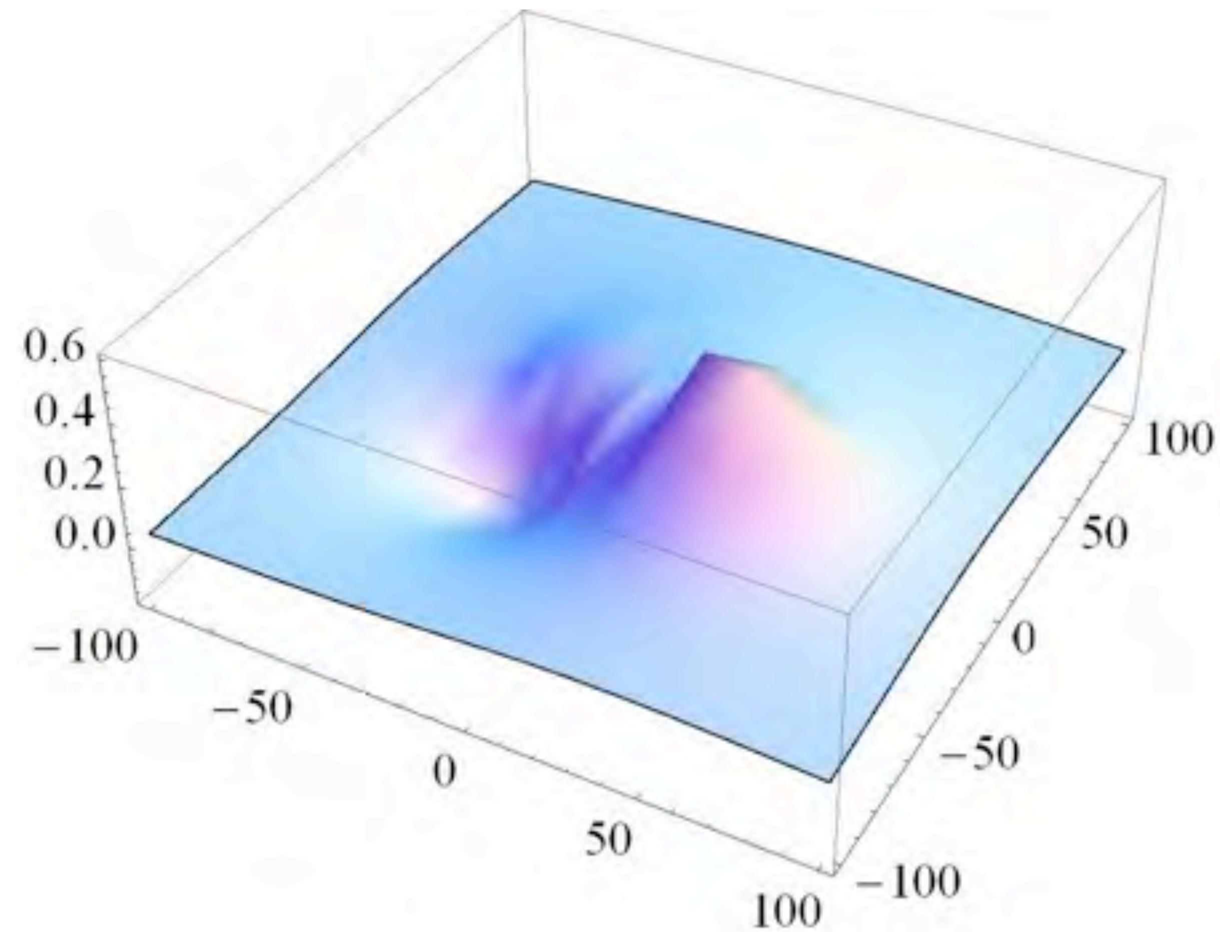




GIS

1. Motivation (cont.)





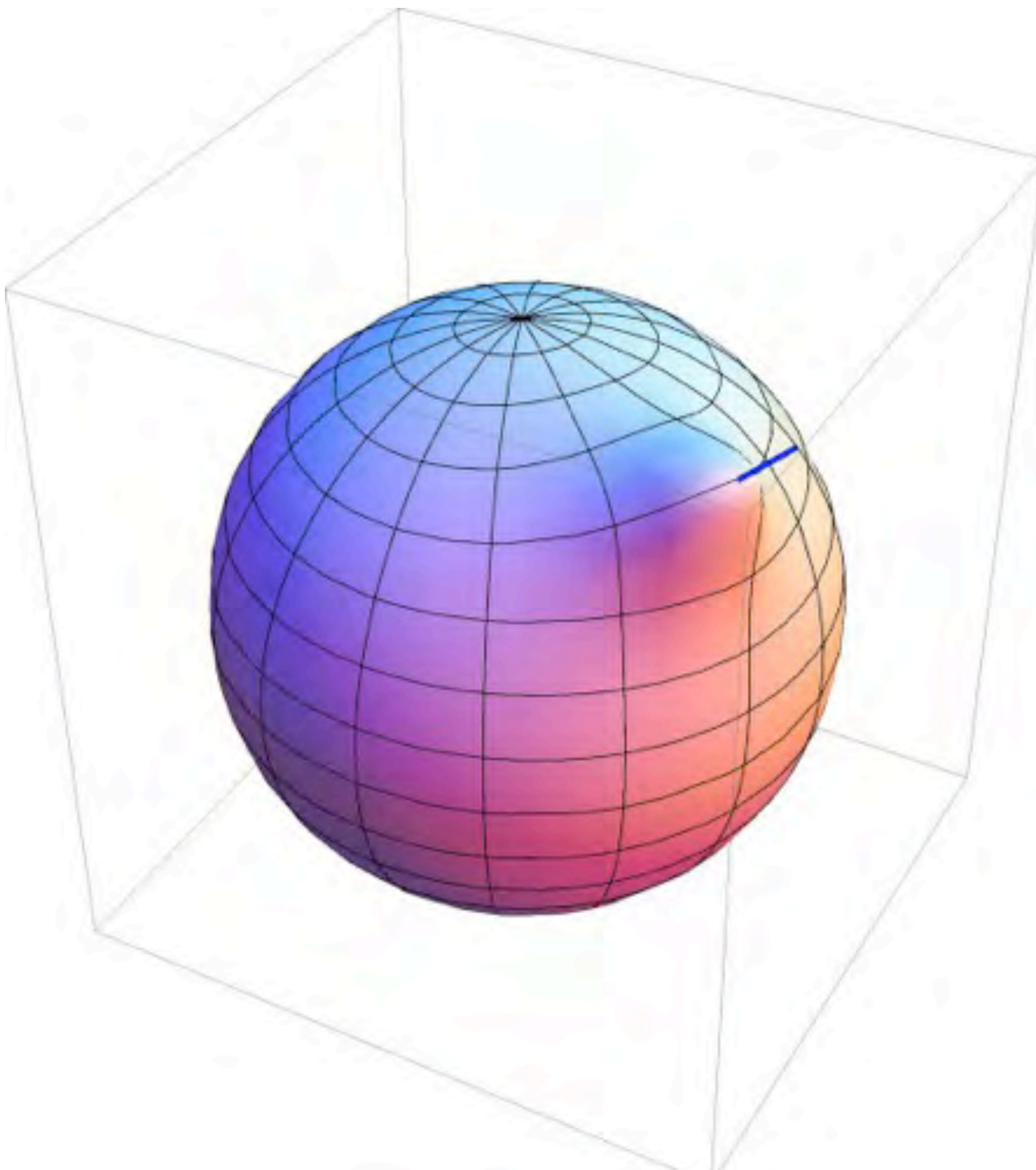
Two options:

identical shape for all base functions
centered at the nodes of a regular grid
=> only the amplitudes have to be optimized
=> linear problem

shapes as well as positions and amplitudes of the base functions are subject to optimization
=> non-linear optimization problem
=> lower number of necessary base functions

3. Radial base functions

A radial base function at the sphere with its center in η , is a function, which is invariant against rotation around a axis passing to η and the center of the sphere.



the value of the base function $\psi(\eta, \bullet)$ in some point ξ on the sphere only depends on the spherical distance between ξ and η .

Hence, the base function must have the following expansion in Legendre polynomials

$$\psi(\eta, \xi, \sigma) = \sum_{0}^{\infty} \sigma_n P_n(\xi^\top \eta)$$

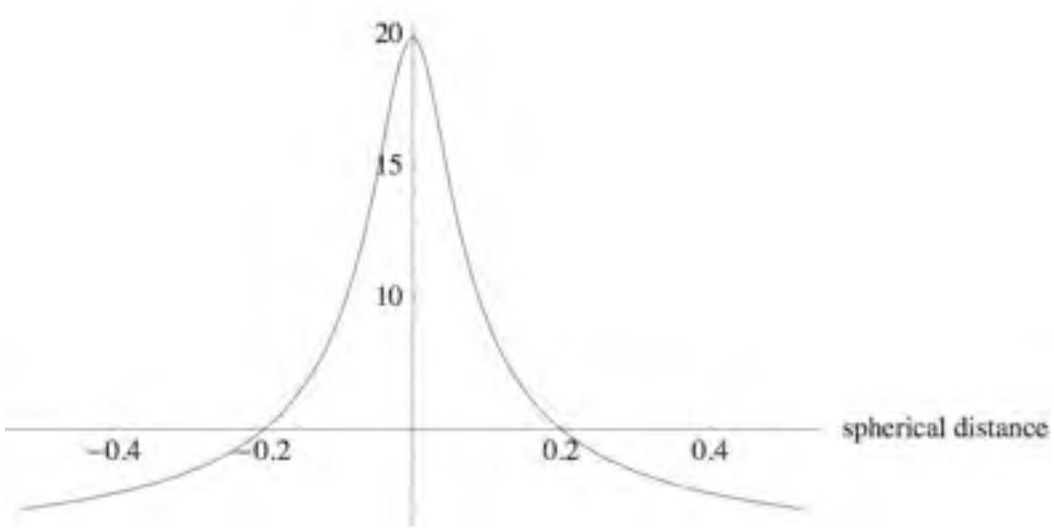
The parameter sequence

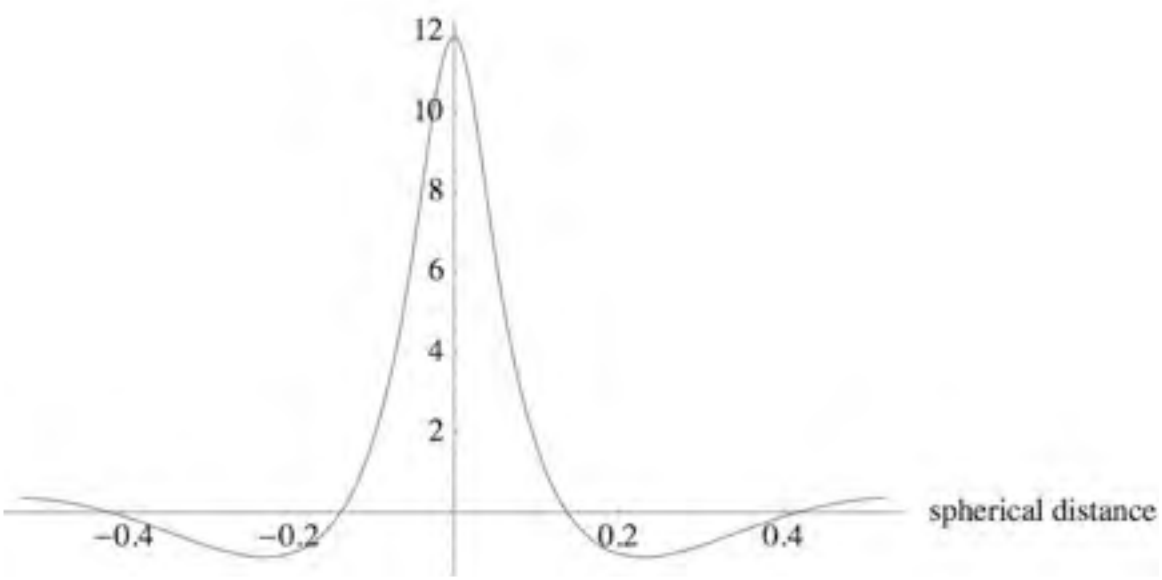
$$\sigma = \{\sigma_0, \sigma, \ldots\}$$

is called shape parameter of the base function.

3. Radial base functions (cont.)

Depending on the choice of the shape parameter, different kinds of base functions are generated





all σ_n positive.
Potential of a buried point mass.

leading σ_n set to zero.
Spherical wavelet.

! in all cases the support of the base function is much smaller than the sphere. Therefore, it is the mathematical description of a local change.

4. Non-linear optimization

Assume the regionally improved gravitational potential be the sum of a global solution V_0 and the sum δV of radial base functions

amplitudes

$$V = V_0 + \delta V = V_0 + \sum_j c_j \psi(\eta_j, \xi, \sigma_j)$$

location parameters

shape parameters

The improved gravitational potential *V* generates via orbit integration synthetic range-rates, which depend upon time and the parameters of the radial base functions:

$$\dot{\rho}_{synth} = \dot{\rho}_{synth}(t, c_j, \sigma_j, \eta_j) = \frac{(\dot{\mathbf{x}}_{2,synth} - \dot{\mathbf{x}}_{1,synth})^\top (\mathbf{x}_{2,synth} - \mathbf{x}_{1,synth})}{\|\mathbf{x}_{2,synth} - \mathbf{x}_{1,synth}\|}$$

The parameters of the radial base function have to be chosen in such a way that at a given number of epochs t_k the synthetic range-rates optimally fit the observed ones:

$$\|\mathbf{Y} - \mathbf{F}(\mathbf{x})\|^2 \rightarrow min$$

$$\mathbf{Y} = (\dot{\rho}(t_1), \ldots, \dot{\rho}(t_N))^\top, \quad \mathbf{F}(\mathbf{x}) = (\dot{\rho}_{synth}(t_1, c_j, \sigma_j, \eta_j), \ldots, \dot{\rho}_{synth}(t_N, c_j, \sigma_j, \eta_j))^\top$$

4. Non-linear optimization (cont.)

The minimum is found using the Levenberg-Marquardt iteration

$$x_{n+1} = x_n + \left((F'(x_n))^\top F'(x_n) + \mu_n I\right)^{-1} (F'(x_n))^\top (Y - F(x_n))$$

$$F'(x_n) = \begin{bmatrix} \frac{\partial\dot\rho}{c_1}(t_1) & \dots & \frac{\partial\dot\rho}{\eta_N}(t_1) \\ \frac{\partial\dot\rho}{c_1}(t_2) & \dots & \frac{\partial\dot\rho}{\eta_N}(t_2) \\ & \vdots & \\ \frac{\partial\dot\rho}{c_1}(t_M) & \dots & \frac{\partial\dot\rho}{\eta_N}(t_M) \end{bmatrix}$$

with

$$\frac{\partial\dot\rho}{\partial p} = \frac{\partial\dot\rho}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial\dot\rho}{\partial\dot x}\frac{\partial\dot x}{\partial p}, \qquad p \in \{c_1, \dots, \eta_N\}$$

Usually, the partial derivatives $\frac{\partial x}{\partial p}, \frac{\partial\dot x}{\partial p}$ are obtained as the solutions of the variational equations

$$\ddot\zeta - \nabla^2 V_0 \zeta = \nabla\delta V, \quad \zeta = \frac{\partial x}{\partial p}$$

4 N systems of ordinary initial value problems of dimension 6 have to be solved in each step

4. Non-linear optimization (cont.)

We aim at a description of the Hessian $\mathbf{F}'(x_n)$ in closed formulas

$$F'(x_n) = \begin{bmatrix} \frac{\partial \dot{\rho}}{c_1}(t_1) & \dots & \frac{\partial \dot{\rho}}{\eta_N}(t_1) \\ \frac{\partial \dot{\rho}}{c_1}(t_2) & \dots & \frac{\partial \dot{\rho}}{\eta_N}(t_2) \\ & \vdots & \\ \frac{\partial \dot{\rho}}{c_1}(t_M) & \dots & \frac{\partial \dot{\rho}}{\eta_N}(t_M) \end{bmatrix}$$

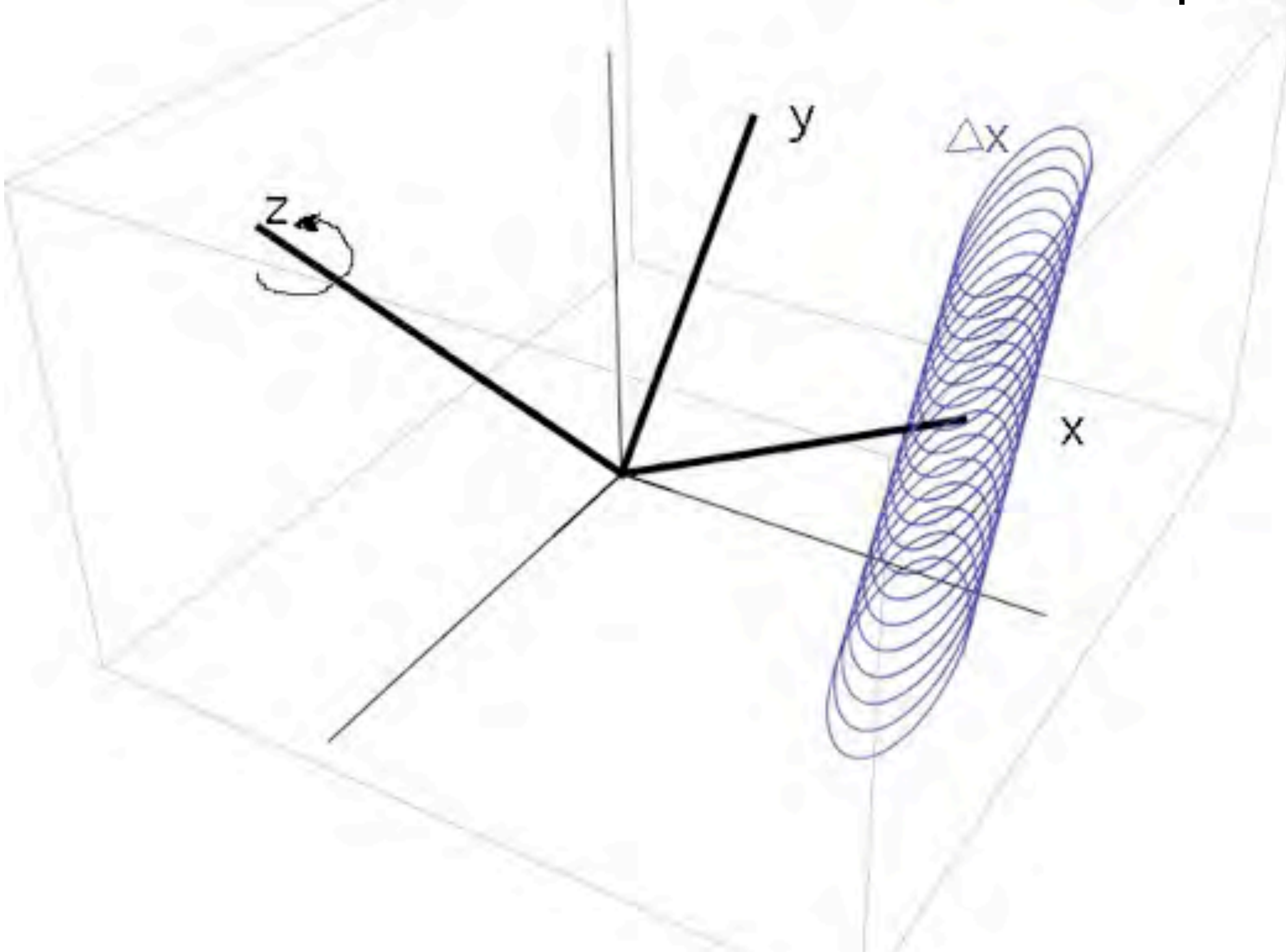
Because, this will not be possible in full generality, we restrict ourselves to short orbital arcs.
For short orbital arcs a simplified orbit model will be sufficiently precise.
One simplified orbital model are the Hill equations of satellite motion.

GIS

5. Hill equations

Assumptions
- almost circular orbit
- arc-length smaller than one revolution

Technique:
- Introduction of an fictitious satellite in the same orbital plane on a circular orbit with the same orbital period as the actual satellite.
- Definition of a rotation coordinate system (x,y,z) with
  - the x-y plane identical to the orbital plane and
  - the x-axis pointing to the artificial satellite.



5. Hill equations (cont.)

The deviations $\Delta\mathbf{x} = (\Delta x, \Delta y, \Delta z)$ of the positions of the actual satellite from the fictitious satellite, with respect to the rotating system solve the following differential equations:

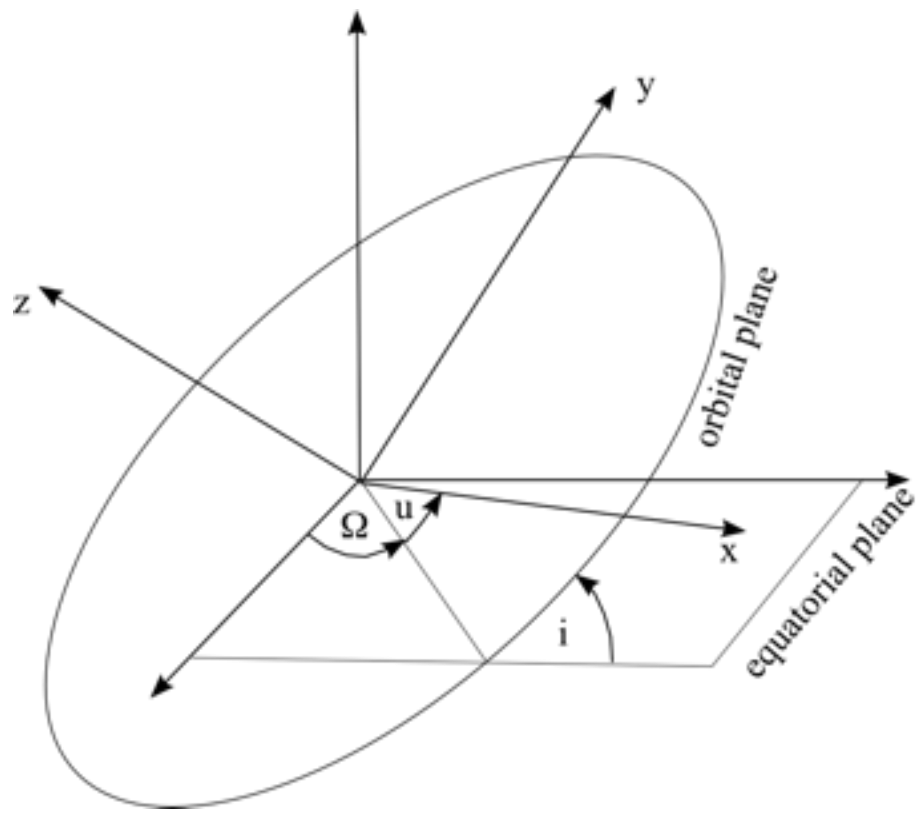
$$\begin{aligned} \Delta\ddot{x} - 2\mu\Delta\dot{y} + 3\mu^2\Delta x &= \frac{\partial V_0 + \delta V}{\partial x} \\ \Delta\ddot{y} + 2\mu\Delta\dot{x} &= \frac{\partial V_0 + \delta V}{\partial y} \\ \Delta\ddot{z} + \mu^2\Delta z &= \frac{\partial V_0 + \delta V}{\partial z} \end{aligned}$$

The corresponding variational equations are

$$\begin{aligned} \Delta\ddot{\zeta}_x - 2\mu\Delta\dot{\zeta}_y + 3\mu^2\Delta\zeta_x &= \frac{\partial^2\delta V}{\partial x\partial p} \\ \Delta\ddot{\zeta}_y + 2\mu\Delta\dot{\zeta}_x &= \frac{\partial^2\delta V}{\partial y\partial p} \\ \Delta\ddot{\zeta}_z + \mu^2\Delta\zeta_z &= \frac{\partial^2\delta V}{\partial z\partial p} \\ (\zeta_x, \zeta_y, \zeta_z) &= (\frac{\partial\Delta x}{\partial p}, \frac{\partial\Delta y}{\partial p}, \frac{\partial\Delta z}{\partial p}) \end{aligned}$$

simple ODE with constant coefficients
=> closed solution can be found, if simple expressions for r.h.s exist

5. Hill equations (cont.)



$$
\begin{aligned}
\Delta\ddot{\zeta}_x - 2\mu\Delta\dot{\zeta}_y + 3\mu^2\Delta\zeta_x &= \frac{\partial^2\delta V}{\partial x\partial p}\\
\Delta\ddot{\zeta}_y + 2\mu\Delta\dot{\zeta}_x &= \frac{\partial^2\delta V}{\partial y\partial p}\\
\Delta\ddot{\zeta}_z + \mu^2\Delta\zeta_z &= \frac{\partial^2\delta V}{\partial z\partial p}\\
(\zeta_x, \zeta_y, \zeta_z) &= (\frac{\partial\Delta x}{\partial p}, \frac{\partial\Delta y}{\partial p}, \frac{\partial\Delta z}{\partial p})
\end{aligned}
$$

For the solution of the variational equations the potential δV produced by radial base functions hast to be rotated into the Hill system. This rotation is given by

$$\mathbf{R}_3(u)\mathbf{R}_1(i)\mathbf{R}_3(\Omega - \Theta), \quad \Theta = \text{sideral angle}$$

6. SO(3) group

A surface spherical harmonic $\overline{Y}_{l,m}(\theta', \lambda')$ in a rotated system can be expressed by a linear combination of its non-rotated cousins $\overline{Y}_{l,m}(\theta, \lambda)$ as:

$$\overline{Y}_{l,m}(\theta', \lambda') = \sum_{k=-l}^{l} D^l_{k,m}(\alpha, \beta, \gamma)\overline{Y}_{l,k}(\theta, \lambda)$$

The weights $D^l_{k,m}(\alpha, \beta, \gamma)$ of this linear combination are the matrix elements of an invariant representation of SO(3) of order 2l+1

$$\mathbf{D}^l = \begin{pmatrix} D^l_{-l,-l} & D^l_{-l+1,-l} & \cdots & D^l_{l,-l} \\ D^l_{-l,-l+1} & D^l_{-l+1,-l+1} & \cdots & D^l_{l,-l+1} \\ & & \ddots & \\ D^l_{-l,l} & D^l_{-l+1,l} & \cdots & D^l_{l,l} \end{pmatrix}$$

GIS

6. SO(3) group (cont.)

These weights are defined by

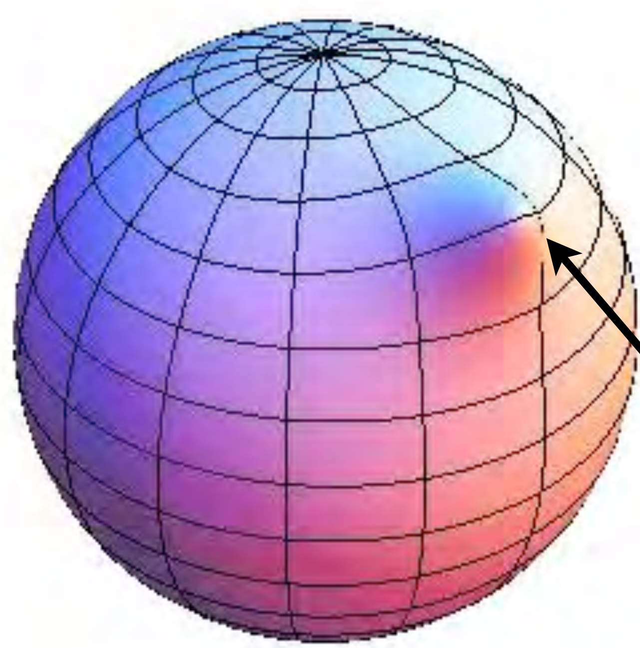
$$D^l_{k,m} = e^{-\imath k\alpha} d^l_{k,m}(\beta) e^{-\imath m\gamma}, \quad d^l_{k,m} - \text{Jacobi polynomials}$$

With the help of the matrix elements of a SO(3) representation spherical harmonics can be transformed into a rotated system.

7. Radial base functions in Hill System

The typical structure of a radial base function with respect to an Earth-fixed system is:

$$\psi(\sigma,\eta,x)=\sum_{n\in N}\sigma^n P_n(\eta^\top x)$$



For a transformation into the rotating Hill System, the following equivalent representation can be used

variable position

$$\psi(\sigma,\eta,x)=\sum_{n\in N}\sigma^n\frac{4\pi}{2n+1}\sum_{m=-n}^{n}\overline{Y_{n,m}}(\eta)Y_{n,m}(x)$$

position of the center with respect to the Earth

Only the second factor has to be transformed into the Hill System

6. Radial base functions in Hill System (cont.)

Transformation of a spherical surface harmonics into a rotated system can be obtained using the representation coefficients of the SO(3) group

$$Y_{n,m}(x) = \sum_{k=-n}^{n} \underbrace{d^n_{k,m}(i) e^{\imath((k-m)\frac{\pi}{2}+m\Omega)} Y_{n,k}(\frac{\pi}{2},0)}_{constant} \cdot \underbrace{e^{\imath(ku-m\Theta)}}_{periodic}$$

Wigner function of inclination

structure of a base function in the Hill System

$$\begin{aligned}\psi(\sigma,\eta,x) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{k=-n}^{n} \underbrace{\frac{4\pi\sigma^n}{2n+1}\overline{Y_{n,m}}(\eta) d^n_{k,m}(i) e^{\imath((k-m)\frac{\pi}{2}+m\Omega)} Y_{n,k}(\frac{\pi}{2},0)}_{=:A_{n,m,k}} \cdot \underbrace{e^{\imath(ku-m\Theta)}}_{periodic} \\ &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{k=-n}^{n} A_{n,m,k} e^{\imath(ku-m\Theta)}\end{aligned}$$

periodic force function in the Hill equations

7. Closed solution of variational equations in Hill system

Recall the variational equations in Hill system

$$
\begin{aligned}
\Delta\ddot{\zeta}_x - 2\mu\Delta\dot{\zeta}_y + 3\mu^2\Delta\zeta_x &= \frac{\partial^2\delta V}{\partial x\partial p} \\
\Delta\ddot{\zeta}_y + 2\mu\Delta\dot{\zeta}_x &= \frac{\partial^2\delta V}{\partial y\partial p} \\
\Delta\ddot{\zeta}_z + \mu^2\Delta\zeta_z &= \frac{\partial^2\delta V}{\partial z\partial p} \\
(\zeta_x, \zeta_y, \zeta_z) &= (\frac{\partial\Delta x}{\partial p}, \frac{\partial\Delta y}{\partial p}, \frac{\partial\Delta z}{\partial p})
\end{aligned}
$$

and the structure of the disturbing force:

$$
\begin{bmatrix} \frac{\partial^2\delta V}{\partial x\partial p} \\ \frac{\partial^2\delta V}{\partial y\partial p} \\ \frac{\partial^2\delta V}{\partial z\partial p} \end{bmatrix} = \sum_j \sum_n \sum_m \sum_k c_j \begin{bmatrix} A^x_{n,m,k} \\ A^y_{n,m,k} \\ A^z_{n,m,k} \end{bmatrix} \cdot e^{\imath(ku-m\Theta)}
$$

$\Rightarrow$ due to the superposition principle for linear ODEs, only the force term $\mathbf{C}e^{\imath(ku-m\Theta)}$ has to be considered

GIS

7. Closed solution of variational equations in Hill system

$$\begin{aligned}
\Delta\ddot{\zeta}_x - 2\mu\Delta\dot{\zeta}_y + 3\mu^2\Delta\zeta_x &= C^x e^{\imath(ku-m\Theta)} \\
\Delta\ddot{\zeta}_y + 2\mu\Delta\dot{\zeta}_x &= C^y e^{\imath(ku-m\Theta)} \\
\Delta\ddot{\zeta}_z + \mu^2\Delta\zeta_z &= C^z e^{\imath(ku-m\Theta)}
\end{aligned}$$

This is the equation of a coupled harmonic oscillator with periodic excitation.
Closed solution can easily be obtained via Laplace transform

The result is the sum of two oscillations:
1. the satellite rotation rate μ
2. the linear combination of the satellite rotation rate μ and the Earth rotation rate $\dot{\theta}$

All together this yields the following computation strategy:

- Computation of the Wigner function only once
- Update of the coefficients ***C*** in each iteration step (due to changing values of p)
- Multiplication with exponentials at each epoch

8. Numerical verification

Aim: Closed formulas for the Jacobian
=
numerical solutions of variational equations

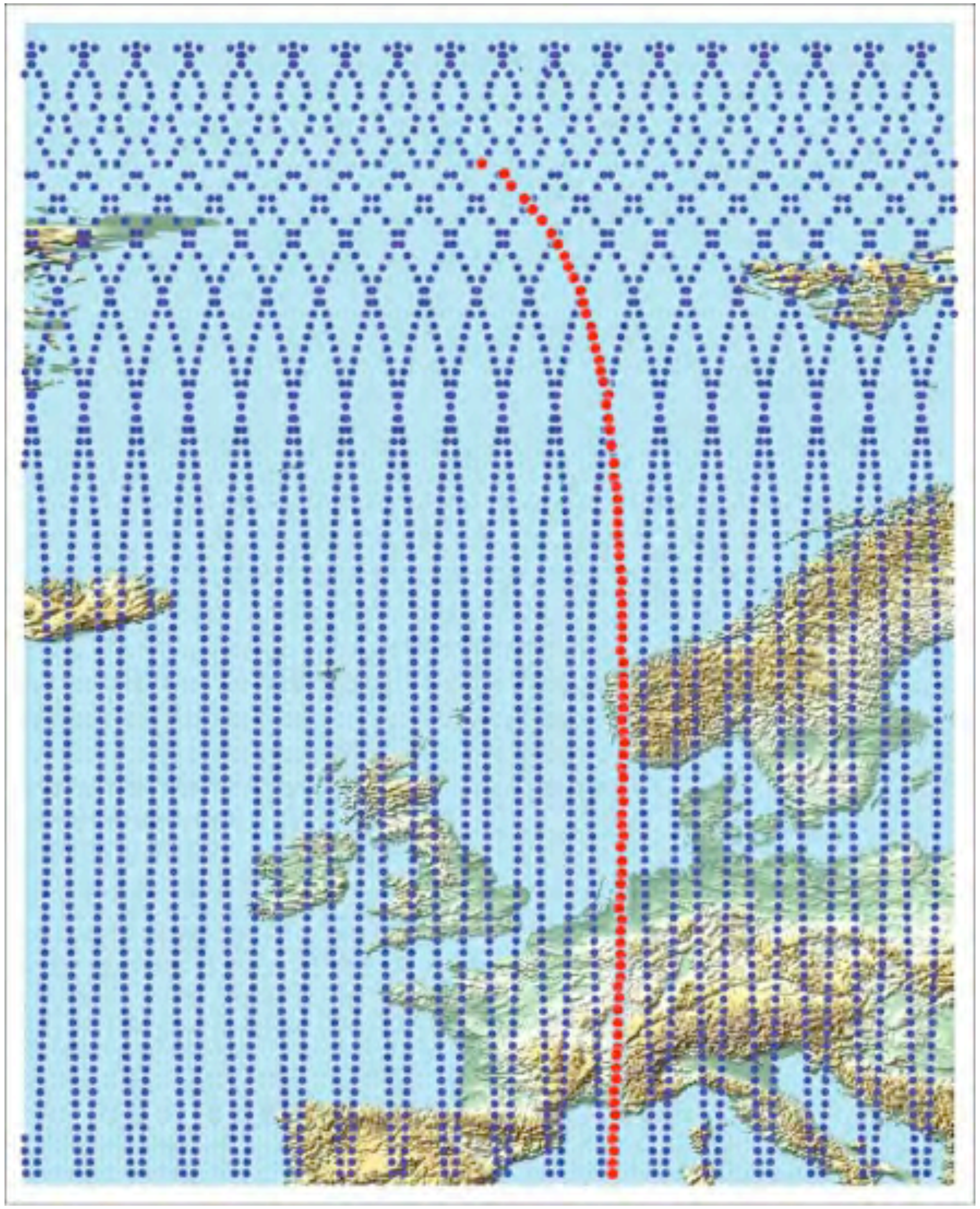
The following August 2002 GRACE scenario was chosen:

	satellite 1	satellite 2
a	6867504 m	6867504 m
e	0,004	0,004
i	89.0169°	89.0169°
Ω	-23.471°	-23.471°
omega	92.861°	93.952°

GIS

8. Numerical verification (cont.)

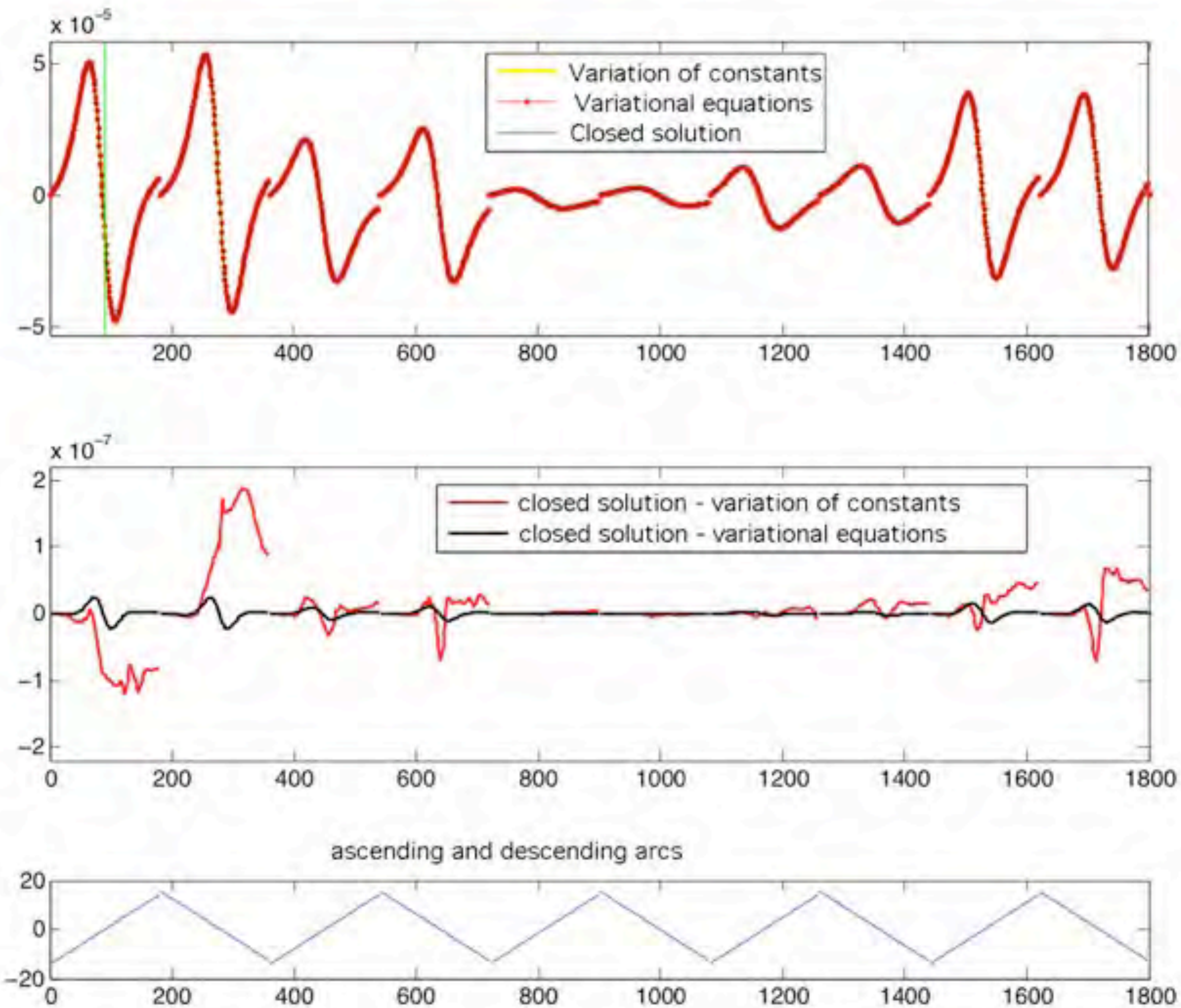
All tracks crossing the following test area were investigated.



investigated base function close to the middle of the arc in red

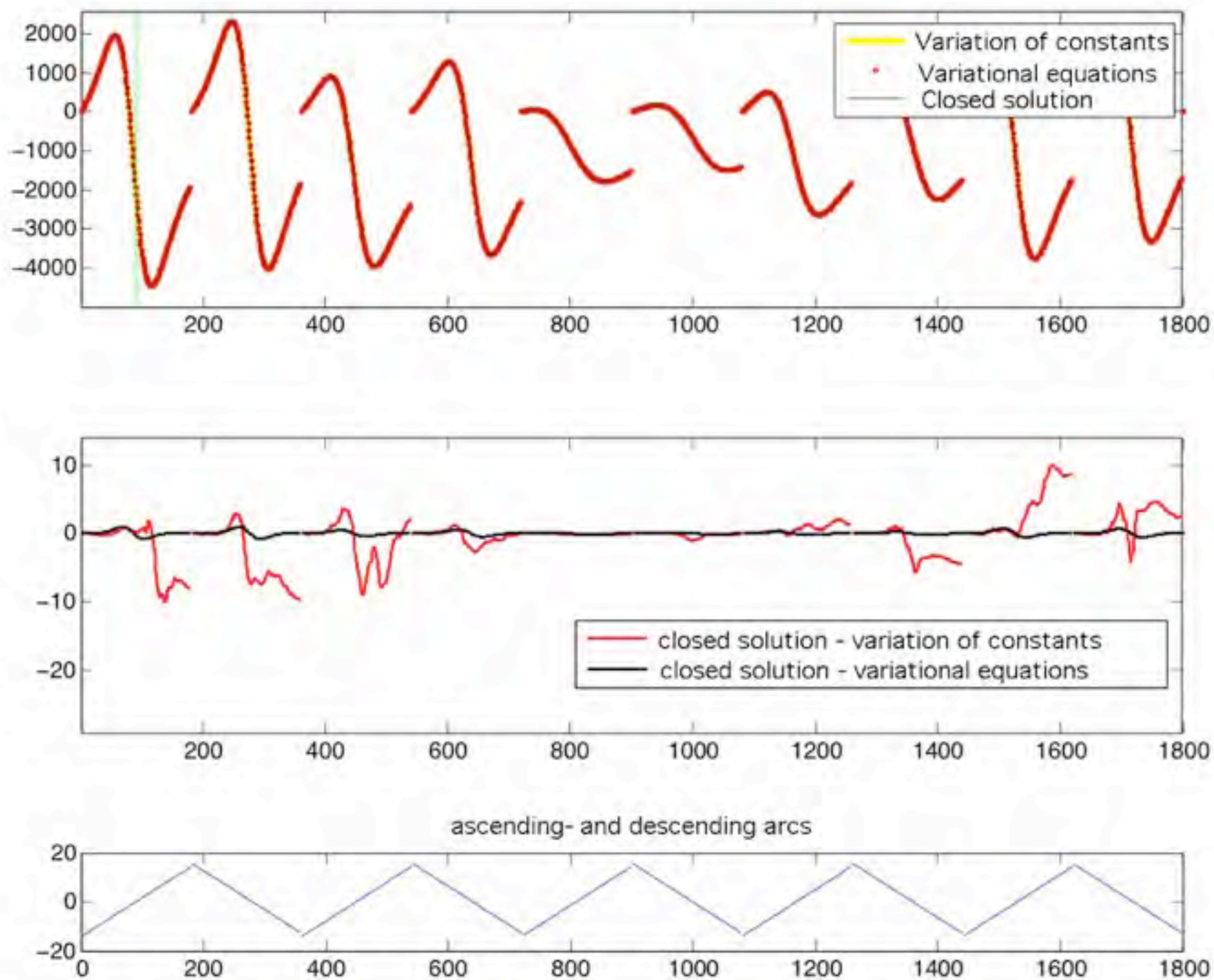
8. Numerical verification

partial derivatives with respect to sigma_j (shape)



8. Numerical verification (cont.)

partial derivatives with respect to c_j (amplitude)



8. Numerical verification (cont.)

Using the closed solution for the variational equations, a small-scale simulation study was carried out. It consisted of four steps:

1. Three radial base functions were randomly chosen. (All parameters randomly). These base function were superimposed the GGM02 field.

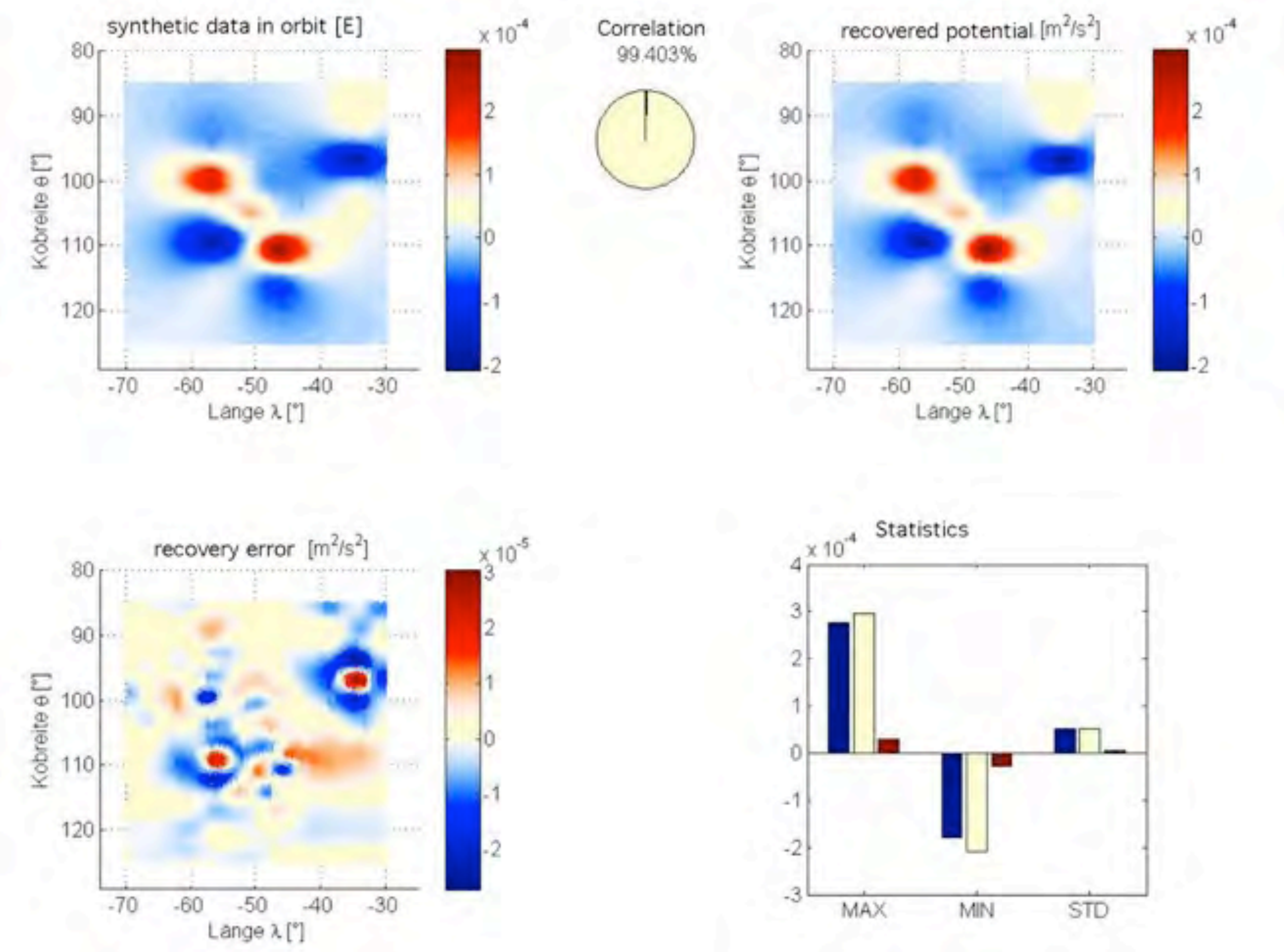
2.The orbits of the two GRACE satellites and the resulting range rates were computed in this combined field.

3. The same procedure was repeated for the GGM02 field alone.

4.From the residual range rates the parameters of the base functions were estimated, using non-linear optimization techniques

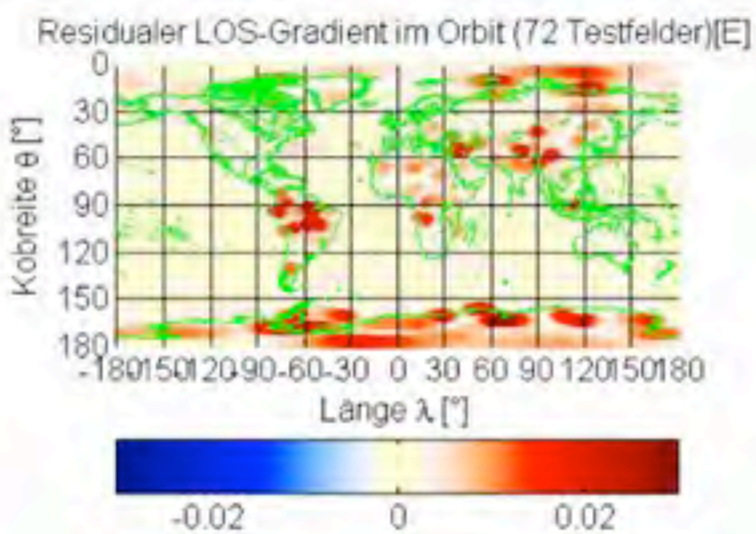
8. Numerical verification (cont.)

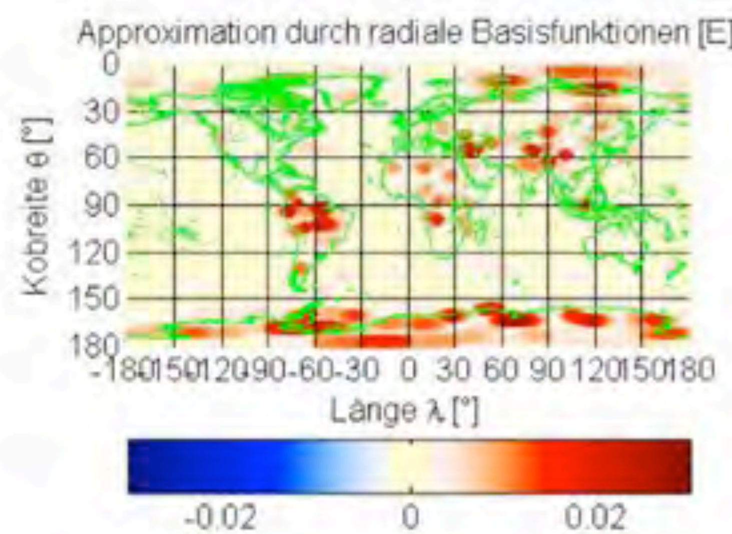
Regional test: (synthetic data)
(i) Collect all data along the orbital arcs.
(ii) Find best fitting radial base function approximation

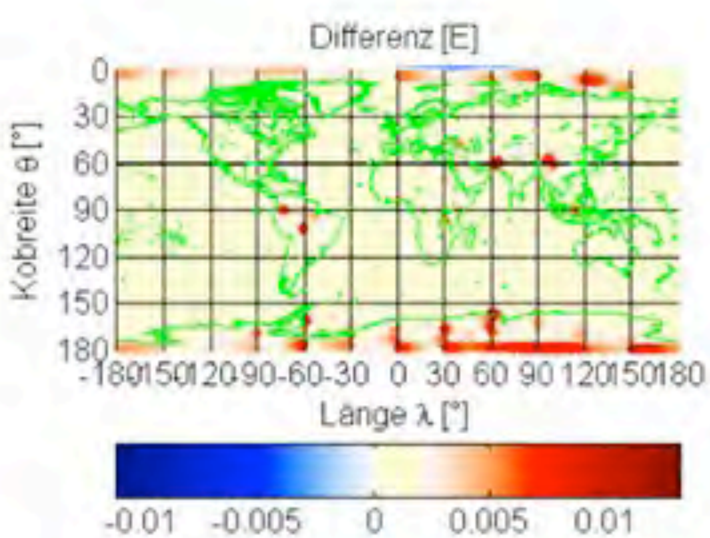


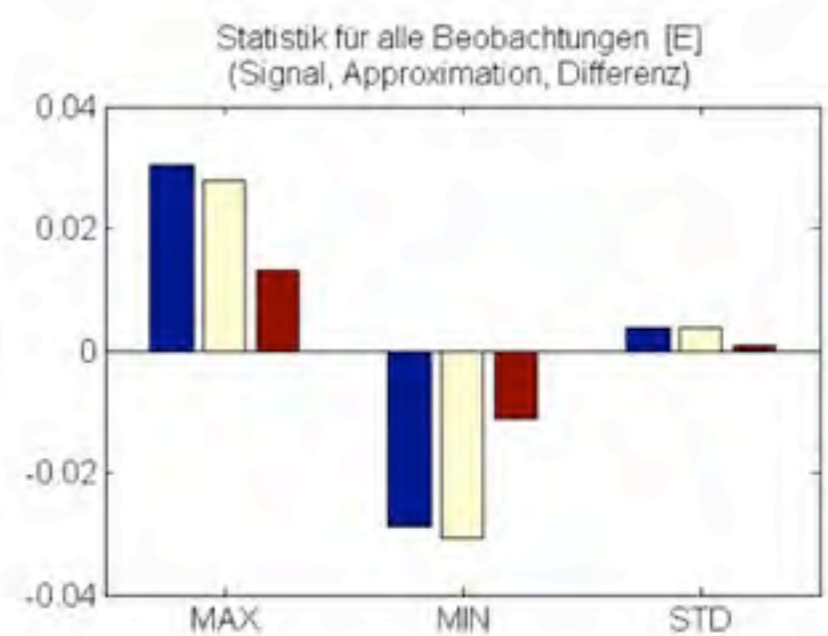
8. Numerical verification (cont.)

Global test:
*GRACE data of August 2002 - GGM02c produce residual observations.
*Earth divided in 72 equi-angular subregions.
*Separate analysis of the regional residual data on each patch
*Merging the 72 patches to a global solution









9. Summary

• approximation of a residual potential by radial base functions with variable sizes and position leads to a non-linear optimization problem

• iterative solution of the non-linear optimization problem requires the computation of the partial derivatives in each step

• the usual computation via variational equation is computational demanding

• an arc-wise treatment in a rotating Hill system leads to variational equations which can be solved in a closed form

• the feasibility of regional gravity field improvement via the closed solution was demonstrated

• further examples for
  - energy-balance approach
  - line-of-sight gradiometry
  - satellite-to-satellite tracking
  - gradiometry

can be found in the PhD thesis of M. Antoni.