Comparing the local gravity field recovery based on radial base functions with the boundary element method

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Motivation

- Residual signal
Motivation

• Leakage – problem:

No function can be space-limited and band-limited at the same time

• Example:
  – Total mass change in equivalent water height
  – CSR GRACE-solutions for a six year period
  – Gauss filter with radius 500km.

Courtesy of Oli Baur, Geodetic Institute Stuttgart
Methodology

- Position-optimized Radial Base Functions
- Boundary Element Method
Methodology

- Position-optimized Radial Base Functions
- Boundary Element Method
Position-optimized Radial Base Functions

Modelling the (residual) signal by superposition of localizing radial base functions:

$$\delta V (\lambda, \theta, r) = \frac{GM}{R} \sum_{b=1}^{B} \eta_b \Psi (\sigma_b, \varpi_b, r)$$

$$= \frac{GM}{R} \sum_{b=1}^{B} \eta_b \sum_{n=1}^{N} \left( \frac{R}{r} \right)^{n+1} \sigma_b (n) P_n (\cos \varpi_b)$$

with:

- $\eta_b$ scale factor
- $\sigma_b (n)$ shape parameter
- $\varpi_b$ spherical distance to the center of the base function
- $P_n (\cos \varpi_b)$ Legendre polynomial
- $\lambda, \theta, r$ spherical coordinates of the point of interest
- $GM$ gravitational constant
- $R$ Earth radius
Properties

- localizing system of base functions
- isotropic = symmetric to the center point
- parameter $\sigma_b(n)$ defines shape
Position-optimized Radial Base Functions

Iterative base search and position optimization

- Reference field
- Preprocessed data
- Initial shape parameter

Nonlinear optimization

- Residual field
- Initial position

Parameter check

- LSQ: scale

Optimized base functions

- LSQ: scale

Initial parameter check
Methodology

- Position-optimized Radial Base Functions
- Boundary Element Method
Motivation of Boundary Element Method

- **Mascon** – approach by Lemoine et al. (2007), Rowlands et al. (2007)
  - successful modelling of GRACE monthly variations
  - use of a small additional layer

\[
\Delta A_{lm}(t) = \frac{(1 + k'_l) R^2 \sigma(t)}{M (2l + 1)} \int Y_{lm} d\Omega
\]

- use of partial derivatives w.r.t. SH-coefficients:

\[
\frac{\partial x}{\partial \sigma_i} = \sum_{lm} \frac{\partial x}{\partial \Delta C_{lm}} \frac{\partial \Delta C_{lm}}{\partial \sigma_i} + \frac{\partial x}{\partial \Delta S_{lm}} \frac{\partial \Delta S_{lm}}{\partial \sigma_i}
\]

- **Possible improvements:**
  - use \( \frac{\partial x}{\partial \sigma_i} \) directly
  - use elements with a finite support

- **Here:** test the approximation quality of different shapes

Boundary Element Method

- Modelling the potential of a single layer

\[ V(\mathbf{x}) = \int_{\Omega} \frac{\sigma(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|} d\Omega \]

- Decomposing the boundary into finite elements:

\[ \Omega = \bigcup_{i=1}^{N} \Omega_i \]

- Assuming a constant behavior of surface mass densities within an element

\[ \sigma|_{\Omega_i} = \sigma_i = \text{const.} \]

\[ V(\mathbf{x}) = \sum_{i=1}^{N} \sigma_i \int_{\Omega_i} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} d\Omega_i \]
Boundary Element Method - Rectangles

- Considering regular rectangles:

\[ \Omega_i = \{(\lambda, \phi) | \lambda_i \leq \lambda \leq \lambda_i + \Delta\lambda_i, \phi_i \leq \phi \leq \phi_i + \Delta\phi_i\} \]

- Potential:

\[
V(x) = \sum_{i=1}^{N} \sigma_i \int_{\Omega_i} \frac{1}{\|x - y\|} \, d\Omega_i
\]

\[
= \sum_{i=1}^{N} \sigma_i \int_{\lambda_i}^{\lambda_i+\Delta\lambda_i} \int_{\phi_i}^{\phi_i+\Delta\phi_i} \frac{R^2 \cos \phi \, d\phi \, d\lambda_i}{\|x - (R \cos \phi \cos \lambda, R \cos \phi \sin \lambda, R \cos \phi)^T \|}
\]

- Discontinuous and non-differentiable elements
- Numerical quadrature
- Many (small) elements for smooth surfaces ⇒ Regularization
Boundary Element Method - Rectangles

- Example for rectangles
Boundary Element Method - Triangles

- Considering triangles and linear interpolation of the surface mass densities and the kernel within a triangle

\[ \kappa_i (x, \lambda, \phi) = \frac{\sigma_{i,1}}{\|x-y(\phi_{i,1}, \lambda_{i,1})\|} \Phi_{i,1} + \frac{\sigma_{i,2}}{\|x-y(\phi_{i,2}, \lambda_{i,2})\|} \Phi_{i,2} + \frac{\sigma_{i,3}}{\|x-y(\phi_{i,3}, \lambda_{i,3})\|} \Phi_{i,3} \]

- Potential:
\[ V(x) = \sum_{i=1}^{N} \int_{\Omega_i} \frac{\sigma_i}{\|x-y_i\|} d\Omega_i \]
\[ = \sum_{i=1}^{N} \sum_{k=1}^{3} \frac{\sigma_k}{\|x-y(\lambda_k, \phi_k)\|} \int_{0}^{1} \int_{0}^{1-\xi} \Phi_{ik} |J| d\eta d\xi \]

- with
\[ \Phi_{i,1} (\lambda (\xi, \eta), \phi (\xi, \eta)) = 1 - \xi - \eta \]
\[ \Phi_{i,2} (\lambda (\xi, \eta), \phi (\xi, \eta)) = \xi \]
\[ \Phi_{i,3} (\lambda (\xi, \eta), \phi (\xi, \eta)) = \eta \]

- Continuous but non-differentiable elements
- Analytical solution of the normal triangle
Boundary Element Method - Triangles

- Example for triangles
Simulation study

a) Single point mass

b) Multiple point masses forming a residual field
Simulation study

a) Single point mass

b) Multiple point masses forming a residual field
a) Single point mass

- Single point mass at depth 125km
- Area: 20° x 20°
- Keplerian orbit
  - height = 385 km
  - 30 days
  - 5 second sampling
  - 3204 observation

- Pseudo-observation: potential energy

\[ V(\lambda, \phi, r) = \frac{2 \cdot 10^{-8} \cdot GM}{\sqrt{(R - d)^2 + r^2 - 2r(R - d)\cos\psi}} \]
a) Single point mass – BEM at depth 10km
a) Single point mass
a) Single point mass – BEM at depth 110km
a) Single point mass - BEM at depth 110km

<table>
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<tr>
<th>Model</th>
<th>RMS  [m²/s²]</th>
<th>Rel. %</th>
<th>Max  [m²/s²]</th>
<th>Rel. %</th>
<th>Min  [m²/s²] rel. %</th>
<th>Corr. %</th>
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<td>0.382</td>
<td>0.61</td>
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<td>0.226</td>
<td>0.36</td>
<td>-0.501</td>
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Statistics:
a) Single point mass - BEM at depth 110km

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<tr>
<th>Method</th>
<th>Number of elements</th>
<th>Regularization</th>
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<tbody>
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<td>BEM (Triangle)</td>
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<td>BEM (Rectangle)</td>
<td>441</td>
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Simulation study

a) Single point mass

b) Multiple point masses forming a residual field
Simulated residual field

- 4225 point masses at depth 120km – 130km
- Area: 20° x 20°
- Keplerian orbit
  - height = 385 km
  - 30 days
  - 5 second sampling
  - 3204 observation

- Pseudo-observation: potential energy

\[ V(\lambda, \phi, r) = \sum_{i=1}^{4225} \frac{\sigma_i \cdot GM}{\sqrt{(R - d_i)^2 + r^2 - 2r(R - d_i) \cos \psi_i}} \]
Simulated residual field - BEM at depth 110km
Simulated residual field - BEM at depth 110km

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Simulated residual field - BEM at depth 110km

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<td>Yes (416)</td>
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Conclusions

• Position-optimized radial base functions for distinct features
  – number of parameter is small (4 x number of bases)
  – problem is non-linear

• Boundary element method for smooth features
  – preferably continuous/differentiable elements (no regularization)
  – grid?
  – preferably numerical quadrature of the Kernel

Outlook:

• Integration: near-zone and far-zone
  – singular, quasi-singular, regular

• Shape elements: higher order triangles and quadrilaterals

• Partial derivatives of the range rate w.r.t. to the surface mass densities
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