1 Introduction

The sensors onboard the GOCE satellite will collect hundreds of millions of observations during the 12 month mission period. The observations are used to determine the Earth’s gravity field, which will be described by thousands of parameters. Therefore, in the case of the GOCE mission the sheer number of observations and parameters makes the processing of the data a challenge for scientists and software engineers. In order to overcome the computational efforts supercomputers are used and parallel computing techniques come into play.

The satellite gravity gradiometer (SGG) is the core sensor onboard GOCE. It is extremely sensitive to high frequencies. What makes the processing of the SGG observations so difficult, is the fact that the SGG observations are highly correlated. For this reason tailored algorithms were derived for the processing of the SGG observations. For example digital filters are used to decorrelate the SGG observations. However, in the presence of data gaps, the application of the algorithms is delicate. The need arises to extend the algorithms to be able to safely handle data gaps. These data gaps can be roughly divided into short and long data gaps, which require different treatment.

One special kind of data gap is the polar cap. It is not a data gap in a strict sense. Due to the sun-synchronous orbit of the satellite, the data does not cover the polar regions. There exist numerous ways to approach this problem. The most straightforward way would be to incorporate additional measurements over the polar regions. Another way is to introduce prior knowledge about the gravity field, which is known as regularization. Finally, one can choose a different set of base functions, known as Slepian’s, which are not defined over certain regions.

2 Gravity field recovery from GOCE data

The gravity field recovery is performed by a least-square adjustment of the observations. Herein, the observation vector \( \mathbf{l} \) is expressed by the product of the design matrix \( \mathbf{A} \) with the parameter vector \( \mathbf{x} \). The design matrix contains the base functions while the parameter vector consists of the gravity field parameters. Because of measurement errors, the resulting system of observation equations is inconsistent. Therefore, the residual vector \( \mathbf{v} \) is introduced to compensate these errors.

\[
\mathbf{l} + \mathbf{v} = \mathbf{Ax}
\]

The base functions are typically spherical harmonics. Thus, the gravity field parameters are typically the spherical harmonic coefficients. As the observations can be considered as a stationary time series along the satellite’s orbit, digital filters can be used to decorrelate the observations. Such a decorrelation corresponds to a multiplication of the observation equations by a lower triangular Toeplitz matrix \( \mathbf{F} \), which we call filter matrix.

\[
\mathbf{F} \mathbf{l} + \mathbf{F} \mathbf{v} = \mathbf{F} \mathbf{A} \mathbf{x}
\]

Each line of the filter matrix \( \mathbf{F} \) can be interpreted as a moving-average filter. The least-squares estimates of the parameters are then obtained by minimizing the residual sum of squares.

\[
\mathbf{v}^T \mathbf{F}^T \mathbf{F} \mathbf{v} \rightarrow \min.
\]

The covariance matrix \( \mathbf{C} \) of the observations is connected to the filter by the following formula.

\[
\mathbf{C}^{-1} = \mathbf{F}^T \mathbf{F}
\]

3 Data gaps

The filtering of the observation equations can be considered as an approach to take the high correlations of the SGG observations into account. For the filtering it is assumed that the observation time series is uninterrupted, which is violated if the observation time series contains data gaps. In this case we need to extend the processing strategy in order to be able to process observations time series.

Data gaps can be roughly divided into long and short data gaps, which require a different treatment. Before a long data gap, the decorrelating filter is stopped and restarted after the gap. This way any correlations between the observations before and after the gap are neglected. For short data gaps, fill-in values are computed such that the filter can run over the gap. After the filtering the fill-in values are removed. This way the correlations between the observations before and after the gap are taken into account.

Figure 1: The processing strategy for long and short data gaps is different. Long data gaps require a restart of the filter after the gaps. Short data gaps require the computation of fill-in values for the data gaps, which are removed after the filtering.
4 Filter warm-up and long data gaps

The filter warm-up is an effect which arises at the beginning of the filtering, i.e. at the beginning of the observation time series and after each long data gap. The problem is that the filtered observations during the warm-up of the filter should not be used within the least-squares adjustment. This way, one can lose many observations due to the filter warm-up if many long data gaps are present in the observation time series. Therefore, a procedure to avoid the filter warm-up has been developed, which is presented in the following. In order to analyse the problem we look at filter matrix $F$. Each row of the filter matrix is in principle a moving-average filter. Because the filters are stable, the coefficients of the moving-average filters converge to zero the farther they are away from the main diagonal. Therefore, in addition to its lower triangular Toeplitz structure the filter matrix can be approximated be a banded matrix. In the upper triangle of the band the moving-average filters are truncated. The more filter coefficients are truncated, the greater is the change in the filter characteristics. This change of the filter characteristics affects only the first filtered observations after the filter start and is the reason for the filter warm-up.

![Figure 2](image)

**Figure 2:** The filter matrix is a lower triangular Toeplitz matrix. In addition it can be approximated by a banded matrix. Each row of the filter matrix is a moving-average filter. In the upper triangle of the band, these filters are truncated, which causes the filter warm-up.

The solution of this problem is to replace the truncated filters by shorter ones which fit in the row and approximate the filter characteristics as good as possible.  

![Figure 3](image)

**Figure 3:** The first part of the filter matrix is replaced in order to avoid the filter warm-up. The computation of the short filters is subject to the Levinson-Durbin algorithm.

This approximation has to be performed with respect to the decorrelating capabilities of the filter. Thus, the autocovariance function of the original filter has to be approximated by the shorter moving-average filters. The well-known Levinson-Durbin algorithm is tailored for the computation of the shorter filters. Within the algorithm moving-average filters from order zero to a desired order are recursively computed. Thus, in each step of the recursion, the algorithm provides one row of the filter matrix.

In order to analyse the performance of this new method the following test has been performed. A correlated time series has been filtered as a reference. In the middle of the time series the same filter has been started one time by the conventional method and one time by the new method.

![Figure 4](image)

**Figure 4:** Setup for the comparison of the conventional filter start which causes a warm-up and the new method for the filter start.

The filter characteristics of the filter which has been used for the analysis of the performance of the new method for the warm-up are given in figure 5. It shows the power spectral density of the filter which is the Fourier transform of the autocovariance function. Thus, it describes the decorrelating capabilities of the filter.

![Figure 5](image)

**Figure 5:** The power spectral density of the filter which has been used for the analysis of the performance of the new method for the filter warm-up.

The impact on the filtered residuals is depicted in figure 6. The filtered residuals of the conventional method are clearly not decorrelated during the warm-up after the filter start. In contrast, the filtered residuals of the new method are very well decorrelated.

![Figure 6](image)

**Figure 6:** Impact of the new method for the filter warm-up on the filtered residual time series. The conventional filter produces an overshoot at the beginning of the filtering. This phenomenon is called the filter warm-up. During the filter warm-up, the filtered residuals apparently do not correspond to a white noise time series. The new method for the start of the filtering prevents the filter warm-up.
The effect of the warm-up can also be analysed by the investigation of the covariance matrix \( C \) which is computed by the filter matrix. Because we assume that the observations are stationary, the covariance matrix of the observations has a Toeplitz structure. Figure 7 shows two covariance matrices. One is computed by the filter matrix of the conventional filter start and the other is computed by the filter matrix of the new method. Apparently, the covariance matrix of the conventional filter start deviates from the desired Toeplitz structure for the covariances of the first observations. This is the effect of the filter warm-up on the covariance matrix. The covariance matrix of the new method is still not a Toeplitz matrix, but it comes much closer to the desired Toeplitz structure. In fact, the covariances sufficiently reflect the accuracy of the observations.

5 Fill-in values and short data gaps

Short data gaps do not require a restart of the filter. Instead, fill-in values for the observations are computed for the short data gaps such that the filter can run over the gaps without doing damage to the filtered residuals. For the computation of the fill-in values one has to be aware of the observations not only contain a deterministic part, but also a stochastic part, i.e. the gravity field information and the measurement errors. The deterministic part of the observations is represented by the adjusted observations

\[
\tilde{I} = A\tilde{x},
\]

wherein \( \tilde{x} \) denotes the least-squares solution for the parameters \( x \). The stochastic part is represented by the residuals

\[
v = \tilde{I} - I.
\]

The determinant part of the fill-in values can be computed by prior knowledge about the parameters or by interpolation techniques. More difficult is the computation of the deterministic part of the fill-in values. It has to accommodate both the residuals before and after the short data gaps and the filter characteristics. This can be achieved by minimizing

\[
v^T F^T F v \rightarrow \min,
\]

whereas the filter matrix \( F \) is fixed and the missing residuals are treated as parameters. The missing residuals can be isolated by splitting the product \( F v \) into two parts.

\[
F v = F_1 v_1 + F_2 v_2,
\]

Herein, \( v_1 \) contains the residuals which are not missing while \( v_2 \) contains the missing residuals. The matrices \( F_1 \) and \( F_2 \) contain the corresponding columns of matrix \( F \). Then, the function which we want to minimize is expressed in terms of the vectors \( v_1 \) and \( v_2 \).

\[
v^T F^T F v = (F_1 v_1 + F_2 v_2)^T (F_1 v_1 + F_2 v_2)
= v_1^T F_1^T F_1 v_1 + 2 v_1^T F_1^T F_2 v_2 + v_2^T F_2^T F_2 v_2
\]

The minimum is obtained by setting the gradient to zero

\[
2 F_1^T F_1 v_1 + 2 F_2^T F_2 v_2 = 0
\]

and solving the equation for \( v_2 \), which yields

\[
v_2 = \left(F_2^T F_2\right)^{-1} F_2^T F_1 v_1.
\]

This corresponds to a least-squares adjustment. The number of parameters can be very great in the case of many small data gaps. In order to overcome this problem, the normal equations are directly computed and stored in a sparse storage scheme. The storage scheme which best suits the problem is

**Figure 7:** Impact of the new method for the filter warm-up on the covariance matrix of the observations. The left panel shows the conventional covariance matrix of the observations derived from the filter. Clearly, during the warm-up of the filter, the covariance matrix deviates from the desired Toeplitz structure. The right panel shows covariance matrix of the observations derived from the filter for the case that the new method for the warm-up is applied. Though this matrix does not possess a Toeplitz structure, it still comes very close to it. The unit of the elements of the covariance matrices is Eötvös \([E^2 = 10^{-9} \text{ m/s}^2]\).
the shell-oriented storage scheme. In this storage scheme the elements of each column are stored contiguously in a vector, beginning at the first non-zero element and ending at the diagonal element of each column. This storage scheme allows for the in-place computation of the least-square solution via the Cholesky decomposition, forward and backward substitution.

6 Polar Gaps

From the mathematical point of view, global gravity field modelling is typically based on spherical harmonic expansion of the potential function. Legendre functions, and thus surface spherical harmonics, are defined globally and satisfy the orthogonality relations on the sphere. However, GOCE satellite ground tracks leave out a double polar cap with a radius of more than six degrees, cf. figure 9.

Figure 9: GOCE mission configuration. The shaded area indicates the spherical belt of data coverage.

The misfit between data measurements and the geopotential modelling is conventionally treated by augmenting data in the polar regions (spatial stabilization), or tailored regularization in the spectral domain using a priori information in terms of spherical harmonic coefficients.

For solving the polar gap problem we apply the $\alpha$-Weighted BLE (Best Linear Estimation), a uniform

Tykhonov-Phillips regularization (Cai, et al. 2004) subject to

$$\hat{x} = (A^T \Sigma^{-1} A + \alpha R)^{-1} A^T \Sigma^{-1} \mathbf{y}$$

complemented by the Mean Square Error matrix

$$MSE \{\hat{x}\} := E \{(\hat{x} - x) (\hat{x} - x)^T\} = D \{\hat{x}\} + \beta \mathbf{b}^T$$

with the bias vector $\mathbf{b} = E \{\hat{x} - x\}$.

$R$ is the regularization matrix, which is typically symmetric and positive definite. In this context, the optimal determination of the regularization parameter $\alpha$ is of prior importance. It balances the variance $D \{\hat{x}\}$ and the squared bias $\beta \mathbf{b}^T$, cf. figure 10. Cai, et al. (2004) and Cai (2004) developed a method to compute the optimal regularization parameter (or weight factor) $\alpha$ by $A$-optimal design according to

$$\hat{\alpha} = \arg \{\text{tr} MSE \{\hat{x}\} = \min\}.$$
combination of GOCE observables and external data, together with regularization is given by

$$\hat{x} = (A_1^T \Sigma_1 A_1 + A_2^T \Sigma_2 A_2 + \alpha R)^{-1} (A_1^T \Sigma_1 l_1 + A_2^T \Sigma_2 l_2),$$

where subscript 1 represents GOCE observables $l_1$ and subscript 2 represents the external observations $l_2$ with the assumption that both kind of observations are not correlated with each other. The regularization matrix $R$ may be chosen to the Gramian matrix, Kaula matrix or a priori information in the low-order domain.

Alternatively, we investigate the Slepian approach (Slepian 1978) to the GOCE mission configuration (Baur and Sneeuw 2007). The philosophy of the method is not to adapt the observation scenario to the modelling but, instead to adapt the parameterization of the geopotential to the observations. This results in band-limited base functions that are optimally concentrated on the spherical belt of data coverage. As shown in Simons et al. (2006), Slepian functions are uniquely defined in case of an axisymmetric polar gap as outlined in figure 9. Figure 11 displays the spatial concentration of Slepian functions.

![Figure 11: Slepian functions ($m = 0, L = 50$). The first four functions (index 1-4) are mainly concentrated in the polar caps C. From the seventh Slepian function on (index 7-51), the signal content in the polar caps is negligible.](image)

7 Conclusions

Data gaps in the observation time series can be treated by different strategies. For long data gaps a special procedure to start the filtering is very useful to avoid a loss of data due to filtering. Short data gaps can be filled with fill-in values, if these are computed with respect to both the deterministic and stochastic part of the observations. The main difference between the two strategies is, that for long data gaps correlations are neglected while for short data gaps correlations are taken into account. Both method perform very well in the case of highly correlated observations such as the SGG observations of GOCE.

To solve the polar gap problem in terms of gravitational field recovery we incorporate three strategies: (1) augmenting data in the polar regions, (2) introducing a priori information in the low-order domain in combination with A-optimal regularization; and (3) the Slepian approach, characterized by a set of base functions that are defined within the area of GOCE data coverage. Numerical tests show the applicability of the Slepian approach with regard to solvability and stability in the case of polar data gaps.

References


