Correction Sheet

F. Krumm and E. Grafarend: Datum-free Deformation Analysis of ITRF networks.
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<table>
<thead>
<tr>
<th>Page 75$^{14}$, 76$^{1}$, 76$^{6}$, 78$^{4}$, 79$^{12}$, 79$^{23}$</th>
<th>$\mathbb{X}_7(3)$</th>
<th>$\mathbb{C}_7(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page 76$^{3}$, 78$^{4}$, 79$^{14}$,</td>
<td>$\mathbb{P}^3$</td>
<td>$\mathbb{R}^3$</td>
</tr>
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<td>Page 76$^{14}$</td>
<td>$T^\text{IM}p^+, t^\text{IM}p^+$</td>
<td>$T \in \mathbb{R}^+, t \in \mathbb{R}^+$</td>
</tr>
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<td>Page 79$^{20}$</td>
<td>$p^{9 \times 9}$</td>
<td>$\mathbb{R}^{9 \times 9}$</td>
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DATUM-FREE DEFORMATION ANALYSIS OF ITRF NETWORKS

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Key words: deformation analysis, datum, ITRF networks

ABSTRACT. In order to perform a proper deformation analysis of geodetic networks an exact definition of the datum-dependence of deformation measures is indispensable. Based upon the notion of fields of displacement and misplacement and the datum transformation of a geodetic network, four scalar-valued deformation measures are introduced which are equivariant (invariant) under the action of the seven parameter conformal group $\mathbb{X}_7(3)$ in three-dimensional space, also known as the similarity transformation (S-transformation: 3 translation parameters, 3 rotation parameters, 1 scale (dilatation) parameter) at any time instant. Special attention is paid to the Error Propagation Law which relates the variances of the scalar-valued deformation measures to the variance-covariance matrices of the coordinates of the network points from one epoch to another one. In the inference the variances of the deformation measures as nonlinear functionals of the variance-covariances of coordinates between epochs are used as decision criteria for a deformation significance test. The significance test is applied numerically to the ITRF2000 network - epochs 1997 and 1997.2 - of type GPS, VLBI, DORIS and SLR.

0. INTRODUCTION

The deformation analysis of geodetic networks embedded in a three-dimensional Euclidean space is conventionally performed in terms of observational functionals which depend on coordinates in a chosen Terrestrial Reference Frame. Indeed, the coordinate estimates as well as their functionals of type displacement, local strain and local vorticity do not depend only on the network geometry and its change in time, but also on the choice of the particular frame of reference. (Dermanis (1985), Dermanis and Grafarend (1993), Grafarend (1985)). However, it has been emphasized that the local displacement vector, the local strain tensor and the local vorticity vector can only be unbiasedly estimated if the underlying frame of reference has not been changed from one instant of time to another one. Alternatively we may relate any deformation analysis on estimable quantities, or invariants under the conformal
group \( \mathbf{X}_\mathbf{r}(3) \) also called similarity transformation at any time instant. Such a similarity transformation (one scale (dilatation) parameter, three rotation parameter and three translation parameter in \( \mathbb{P}^3 \)) leaves angles and distance ratios invariant (Baarda (1967, 1973), Grafarend and Schaffrin (1976)).

Here we introduce four (additive and multiplicative) choices of deformation measures which are equivariant/invariant under the action of the conformal group \( \mathbf{X}_\mathbf{r}(3) \). In order to infer significant deformations from the estimated deformation measures we compute their variances as derived via Error Propagation from coordinate estimates, the elementary basis of an intuitive significance test. Finally we apply the datum-free deformation analysis to the ITRF2000 networks observed by GPS, VLBI, DORIS and SLR.

1. ESTIMABLE QUANTITIES AND DATUM INVARIANTS IN DEFORMING NETWORKS

Many local, regional and global geodetic networks have been set-up with the aim to monitor deformations, for instance of building blocks, or movements, for example of tectonic plates. Those deforming networks are observed at various time instants and analyzed with respect to the placement of their nodal points coordinated with respect to a particular frame of reference ("datum") at epoch. Conventionally the geodetic networks are adjusted with respect to nodal point coordinates in a Linear Model and tested consequently ("hypothesis test": t-test) with respect to

- no change of coordinates between epochs

 alternatively

- a local change of a particular coordinate set

under the assumption of a given variance-covariance matrix of the observations. In fact, such a procedure is acceptable if it can be guaranteed that the frame of reference has not been changed from one epoch to another. But such an assumption is rarely met in practice.

In terms of a three-dimensional network, a first additive measure of deformation is the displacement vector \( \mathbf{D}\{\mathbf{X}(T),\mathbf{x}(t)\} = \mathbf{x}-\mathbf{X} \) with respect to placement vectors \( \mathbf{X}(T) \) and \( \mathbf{x}(t) \) at epoch \( T \) and \( t>T, T+\mathbf{P}, T+\mathbf{P} \), respectively. Let us assume that the placement vectors \( \mathbf{X}(T) \) and \( \mathbf{x}(t) \) as well as the displacement vector \( \mathbf{D}\{\mathbf{X}(T),\mathbf{x}(t)\} \) is represented in the fixed orthonormal frame \( \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}_O \) at reference epoch \( T \) and origin \( O \). By means of formulae (1)-(4) in Table 1 we summarize such representations in the Lagrange portrait which is also called "material".

However, in the more realistic case that the set of orthonormal base vectors from one epoch to the other has changed, no deformation can be detected on a sound basis. For instance, in the case that a specific nodal point has not moved, the variation of the frame of reference would indicate an artificial displacement. Indeed, we like to describe the difference between two placement vectors \( \mathbf{X}(T) \) and \( \mathbf{x}(t) \), respectively, given on the one side in the frame of reference \( \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}_O \) at epoch \( T \) and origin \( O \) and on the other side in the frame of reference \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}_O \) at epoch \( t \) and origin \( o \) by the term misplacement. Formulae (5)-(8) in Table 2 are representatives of such an unpleasant situation. Repair can be made by shifters (Eringen 1962, pages 9,447) which transform \( \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}_O \) to \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}_O \) and vice versa.
Table 1

Representation of the placement vectors \( \mathbf{X}(T), \mathbf{x}(t) \) as well as the displacement vector \( \mathbf{D}\{\mathbf{X}(T),\mathbf{x}(t)\} = \mathbf{x} - \mathbf{X} \) in the orthonormal frame of reference \( \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3|\mathbf{O}\} \) at reference epoch \( T \) and origin \( \mathbf{O} \).

\[
\mathbf{X}(T) = E_1X^1 + E_2X^2 + E_3X^3 = \sum_{i=1}^{3} E_iX^i
\] (1)

\[
\mathbf{x}(t) = E_1x^1 + E_2x^2 + E_3x^3 = \sum_{i=1}^{3} E_iX^i
\] (2)

\[
\mathbf{D}\{\mathbf{X}(T),\mathbf{x}(t)\} = E_1(x^1 - X^1) + E_2(x^2 - X^2) + E_3(x^3 - X^3) = \sum_{i=1}^{3} E_i(x^1 - X^1)
\] (3)

\[
\langle E_i \mid E_j \rangle = \delta_{ij} = \begin{cases} 
1 & \text{for all } I = J \\
0 & \text{for all } I \neq J 
\end{cases} \quad I,J \in \{1, 2, 3\}
\] (4)

Table 2

Representation of the placement vectors \( \mathbf{X}(T), \mathbf{x}(t) \) as well as the misplacement vector in the orthonormal frames of reference \( \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3|\mathbf{O}\} \) and \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3|\mathbf{O}\} \), respectively.

\[
\mathbf{X}(T) = E_1X^1 + E_2X^2 + E_3X^3 = \sum_{i=1}^{3} E_iX^i
\] (5)

\[
\mathbf{x}(t) = e_1x^1 + e_2x^2 + e_3x^3 = \sum_{i=1}^{3} e_iX^i
\] (6)

\[
\mathbf{x}(t) - \mathbf{X}(T) = \sum_{i=1}^{3} e_iX^i - \sum_{i=1}^{3} E_iX^i
\] (7)

\[
\langle e_i \mid e_j \rangle = \delta_{ij}, \quad \langle e_i \mid e_j \rangle = \delta_{ij} \quad I,J,i,j \in \{1, 2, 3\}
\] (8)

Finally let us introduce the datum-free concept of deformation measures. Denote, according to Figure 1, the triplet of placement vectors \( \{\mathbf{X}_A, \mathbf{X}_B, \mathbf{X}_C\} \) at epoch \( T \) and \( \{\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c\} \) at epoch \( t \) representing three points \( \{\mathbf{P}_A, \mathbf{P}_B, \mathbf{P}_C\} \), \( \{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c\} \), respectively, which form a triangular element \( \Delta\{\mathbf{P}_A, \mathbf{P}_B, \mathbf{P}_C\}, \Delta\{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c\} \), respectively.
There exist four choices (13)-(16) of invariant measures of deformation (see Table 3) which are equivariant under the action of the conformal group $\mathbb{R}^3(3)$ in $\mathbb{R}^3$ at epoch $T$ and epoch $t$, respectively. They are derived either from space angles (position angles) $\cos \Psi_{\text{BAR}}$, $\cos \Psi_{\text{BAr}}$, respectively or from distance ratios $\|X_B - X_A\|^2 / \|X_T - X_A\|^2$, $\|x_B - x_A\|^2 / \|x_T - x_A\|^2$ given by definitions (9)-(12).

\[
\cos \Psi_{\text{BAR}} := \langle X_B - X_A, X_T - X_A \rangle / (\|X_B - X_A\| \|X_T - X_A\|) \tag{9}
\]

\[
\cos \Psi_{\text{BAr}} := \langle x_B - x_A, x_T - x_A \rangle / (\|x_B - x_A\| \|x_T - x_A\|) \tag{10}
\]

\[
\|X_B - X_A\|^2 := (X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)^2 \tag{11}
\]

\[
\|x_B - x_A\|^2 := (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2 \tag{12}
\]

<table>
<thead>
<tr>
<th>Deformation measure I:</th>
<th>Deformation measure II:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of position angles</td>
<td>Difference of position angles</td>
</tr>
<tr>
<td>ratio$\Psi := \cos \Psi_{\text{BAR}} / \cos \Psi_{\text{BAr}}$</td>
<td>diff$\Psi := \cos \Psi_{\text{BAR}} - \cos \Psi_{\text{BAr}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Deformation measure III:</th>
<th>Deformation measure IV:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of distance ratios</td>
<td>Difference of distance ratios</td>
</tr>
<tr>
<td>ratio$\delta := \frac{|X_B - X_A|^2}{|X_T - X_A|^2} = \frac{|X_B - X_A|^2}{|x_B - x_A|^2} \frac{|x_T - x_A|^2}{|x_T - x_A|^2}$</td>
<td>diff$\delta := \frac{|X_B - X_A|^2}{|X_T - X_A|^2} - \frac{|x_B - x_A|^2}{|x_T - x_A|^2}$</td>
</tr>
</tbody>
</table>

Table 3
Datum-free deformation measures
2. THE DEFORMATION SIGNIFICANCE TEST

The deformation significance test is based upon the deviation of the deformation measure from its ideal value taking its variance into account: The ratios should be equal to one, the differences should vanish if the geodetic network does not deform. A test is performed by comparing the deviation from one or zero with some multiple of its standard deviation. Note that for each observation technique there are no redundant observations available which build up an overdetermined system; the investigated networks of type GPS, VLBI, DORIS and SLR have not enough points in common in order to use adjustment techniques and to apply the standard Student t-test to our deformation measures, unfortunately. In the future we hope that this situation will change, namely that estimates of the deformation measures with their empirical variances and covariances are available. In addition, we have to find the sampling distribution of the $X_7(3)$ equivariant deformation measures assuming that the estimated coordinate data at points $\{P_A, P_B, P_\Gamma, P_\alpha, P_\beta, P_\gamma\}$ are normally distributed. Since the scalar-valued deformation measures build up a nonlinear function of the coordinates of the six points in $P^3$, we do not expect the sampling distribution to be of Student type.

With the intention to derive variances as well as standard deviations of the deformation measures I-IV given the variances as well as the covariances of the Cartesian coordinates of the six points $\{P_A, P_B, P_\Gamma, P_\alpha, P_\beta, P_\gamma\}$ we apply the linearized Law of Error Propagation (Graafarend and Schaffrin 1993, page 469) to equations (13)-(16). Given the variance-covariance matrices $\Sigma_X \in P^{9 \times 9}$ and $\Sigma_s \in P^{6 \times 6}$ of the Cartesian coordinates of the three points $\{X_A, X_B, X_\Gamma\}$ as well as $\{x_\alpha, x_\beta, x_\gamma\}$ which build up the triangular element at epoch $T$ and epoch $t$, respectively, and assuming no correlation between epochs, $\Sigma_{X(T)X(t)} = 0$, the variances of the $X_7(3)$ equivariant deformation measures are easily derived.

3. NUMERICAL TESTS

The deformation analysis based upon the deformation measures (13)-(16) has been applied to ITRF2000 networks of epoch 1997 and 1997.2, respectively, split by

- GPS-coordinates
- VLBI-coordinates
- DORIS-coordinates
- SLR-coordinates

(ftp via lareg.ensg.ign.fr/pub/itr4)

For further details we refer to the IERS homepage http://lareg.ensg.ign.fr/ITRF as well as to Boucher et al. (1999) and Sillard et al. (1998). In a preprocessing phase the "deformed" ITRF2000 network (epoch 1997.2) was generated from the ITRF2000 network at epoch 1997.0 using the velocities as provided by the SINEX data files, and the models for station coordinate and variance-covariance propagation, respectively, $x(t) = X(T) + (T-t)v$, $\Sigma_x(t) = \Sigma_{X(T)} + (T-t)(\Sigma_{xx} + \Sigma_{vv}) + (T-t)^2 \Sigma_v$.

The deformation analysis was explicitly started by first deriving a polyhedral triangular network which is both unique and independent on the choice of the analyst. Indeed all points of the individual solutions were projected along the surface normal onto the International Reference Ellipsoid GRS80. Following this, the surface (foot) points on the ellipsoid were mapped onto a plane by taking advantage of the famous ellipsoidal Hammer projection (Graafarend and Syffus 1997) which is equiareal. Finally the set of planar coordinates were connected by a Delaunay triangulation delivering all point combinations which had to form a
unique polyhedral triangular network in the ambient three-dimensional Euclidean space. A Delaunay triangulation of a point set is a triangulation of the point set with the property that no point in the point set falls into the interior of the circumcircle - the circle that passes through all three vertices - of any triangle in the triangulation. The point distributions of the networks as a result from the ellipsoidal Hammer projection are plotted in Figure 2 (GPS, 375 stations), Figure 6 (VLBI, 147 stations), Figure 9 (DORIS, 82 stations) and Figure 12 (SLR, 142 stations).

As a first result of the analysis it can be stated that there is practically no difference between the deformation tests I and II ("ratio and difference of position angles") and only a rather small variation in sensitivity between deformation tests III and IV ("ratio and difference of distance ratios"). In Table 4 we have listed the number of triangles and the percentages of deformed elements as being detected by the kσ-rule. A value of 0% indicates that exactly no deformed triangle has been detected.

<table>
<thead>
<tr>
<th></th>
<th>No. of points</th>
<th>No. of triangular elements</th>
<th>1σ Deformation tests</th>
<th>2σ Deformation tests</th>
<th>3σ Deformation tests</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>I=II III IV</td>
<td>I=II III IV</td>
<td>I=II III IV</td>
</tr>
<tr>
<td>GPS</td>
<td>375</td>
<td>697</td>
<td>4.9% 2.7% 2.4%</td>
<td>1.3% 0.6% 0.6%</td>
<td>1% 0.3% 0.4%</td>
</tr>
<tr>
<td>VLBI</td>
<td>147</td>
<td>254</td>
<td>1.6% 0.0% 0.4%</td>
<td>0.8% 0.0% 0.4%</td>
<td>0.4% 0.0% 0%</td>
</tr>
<tr>
<td>DORIS</td>
<td>82</td>
<td>126</td>
<td>1.6% 0.8% 0%</td>
<td>0.8% 0% 0%</td>
<td>0% 0% 0%</td>
</tr>
<tr>
<td>SLR</td>
<td>142</td>
<td>258</td>
<td>0.4% 0.4% 0%</td>
<td>0% 0% 0%</td>
<td>0% 0% 0%</td>
</tr>
</tbody>
</table>

Concerning the difference between the deformation measures I=II and III,IV the statistics above shows that position angle based deformation measures I,II are more sensitive than distance ratio based deformation measures III,IV. In addition, many deformed triangles which are discovered by deformation test I=II are different from those ones detected by deformation measures III and IV. This can be roughly suspected from Figures 3 and 4 (GPS: deformation measures I and III) or Figures 7 and 8 (VLBI: deformation measures I and IV), or better taken from a comparison of data sets at a longer interval of time, e.g. epochs 1997.0 and 1997.4.

Going into more detail we distinguish between the different observation types/networks of identical points with respect to both epochs.

First the GPS network has the most stations which are very well distributed all over the globe with some slight concentration in Europe and North America. Nevertheless the polyhedral network shows relatively large triangles spanning many plate boundaries. Naturally the number of deformed triangles decreases as the factor k in our deformation tests increases.

The VLBI-network suffers from the fact that the points are located mainly in North America, Europe and Japan. Only a very limited number of points can be found on the southern hemisphere, so that many small sized and a few large sized triangles exist. Clearly, the large triangles span several plate boundaries. It is expected that the portion of deformed large triangles prevail small ones, but this is not the case, however.

Contrary to the VLBI-network the DORIS-network points are distributed very well around the globe. Due to this excellent feature of the network deformed triangles are of moderate size only; exotic sized deformed triangles do not exist in comparison with the VLBI or SLR network. The latter one has almost the same number of points and a similar structure as the VLBI-network: Only a few points are located on the southern hemisphere mainly in regions of mean latitude; no station exists beyond a (southern) latitude of -35°.
4. CONCLUSIONS

First of all it should be stressed that it was not our primary intention to judge the quality of the analysed networks. They were chosen only as an example to demonstrate the proposed deformation tests. From the theoretical point of view, it is logical that deformation tests have to be based on $X_r(3)$-invariant deformation measures, otherwise scientific acknowledgment and integrity cannot be achieved; the best deformation test is nothing worth if a change of the reference frame is either not taken into account or not eliminated by a use of properly defined deformation measures. However, from the numerical test as described above it can be seen that a general statement on their usefulness as applied here is difficult to make. The numerical results obtained so far raise additional questions, e.g. what the role of the triangulation method is like? Naturally, it would be desirable to have sets of equally shaped and equally sized triangles, e.g. in form of a Bruns polyeder around the globe. Unfortunately, real networks do not establish such an ideal network. On the other hand, a triangulation method should be chosen which does not depend on the subjective opinion of the analyst. The Delaunay triangulation fulfills this requirement! Of course a 3D-Delaunay triangulation method/algorithm would be desirable in order to avoid the mapping process, but to our knowledge this is not available, unfortunately. Once again, we want to stress that projecting the points onto an ellipsoid and mapping the foot points using a certain map projection is done only for the sake of finding those points which generate the three dimensional net of spatial triangles. A further question is if there exists any influencing relationship between a deformation of a triangle and its area/size? A first conjecture would say yes, but the numerical/graphical analysis indicates very clearly that this is not the case: All shape and size variations of deformed triangles are equally represented. An important question is the consequence for the nodes if a triangle is displayed as distorted but the neighboring triangles are not? Is there any means to isolate those points which are responsible for an indicated distortion? Both questions can hardly be answered without additional, i.e. redundant information. Our scalar-valued deformation functionals of vector-valued arguments, i.e. placement vectors, can only indicate that "something has happened"; they are not capable to locate the source in terms of a single point. Finally the question arises, how the deformation tests can be improved and be based on a less heuristic approach? This question is easily answered but probably more complicated to be put into practice. First of all, more sites should be equipped with different observation techniques leading to redundant coordinate information and a tighter link between the different network types. Second, global networks should be of regular shape with good point distribution around the globe; the disadvantages as described above for the single networks should be avoided.

ACKNOWLEDGEMENT

The spherical coordinates - converted to ellipsoidal coordinates - of the tectonic plate boundaries have been provided by the Deutsches Geodätisches Forschungsinstitut (DGFI), München.

REFERENCES


Figure 2: The Delaunay triangulation of the global GPS network (ITRF2000)

Figure 3: Deformation measure I (GPS, ITRF2000, 1σ)
Figure 4: Deformation measure III (GPS, ITRF2000, 1σ)

Figure 5: Deformation measure IV (GPS, ITRF2000, 1σ)

Figure 6: The Delaunay triangulation of the global VLBI network (ITRF2000)

Figure 7: Deformation measure I (VLBI, ITRF2000, 1σ)

Figure 8: Deformation measure IV (VLBI, ITRF2000, 1σ)
**Figure 9:** The Delaunay triangulation of the global DORIS network (ITRF2000)

**Figure 10:** Deformation measure I (DORIS, ITRF2000, 1σ)

**Figure 11:** Deformation measure III (DORIS, ITRF2000, 1σ)

**Figure 12:** The Delaunay triangulation of the global SLR network (ITRF2000)

**Figure 13:** Deformation measure I (SLR, ITRF2000, 1σ)

**Figure 14:** Deformation measure III (SLR, ITRF2000, 1σ)

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