A geometric perspective to the boundary value problem of physical geodesy

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Introduction

According to Friedrich Robert Helmert (1843-1917) geodesy aims at the determination of the geometrical shape of the Earth’s surface. Since this is per definitionem a geometric problem one expects that it can be solved by geometric means alone. At first glance, it is not obvious that also physical methods are necessary to solve the problem of the determination of the Earth’s geometric size and shape. This contribution tries to explain the reasons behind the intertwining of geometry and gravity in Physical Geodesy.

Aspects of projective geometry

In a certain sense, projective geometry can be seen as a way out of the fundamental crisis of Euclidean geometry at the end of the 18th century. Euclidean geometry had played a model role for all formal sciences: from a very few axioms, which had been considered obvious and not requesting any proof, the whole building of geometry was constructed by purely logic deduction. The reputation of Euclidean geometry was so big, that the German philosopher Immanuel Kant (1724-1804) declared it as an “a priori condition of our internal intuition”. The only nuisance was the postulate of parallels:

To a given straight line \( g \) in the plane and a point \( P \) outside of \( g \), there is exactly one other straight line \( g' \), which passes \( P \) and does not intersect \( g \).

This axiom has two peculiarities: firstly, most of the elementary results in Euclidean geometry are symmetric, e.g. “A triangle with equal sides has equal angles” and “A triangle with equal angles has equal sides”. The postulate of parallel destroys this symmetry even for the fundamental axioms. One of the axioms is:

Two points \( P \) and \( P' \) are connected by exactly one straight line.

Due to the postulate of parallels its symmetric counterpart

Two straight lines \( g \) and \( g' \) intersect in exactly one point \( P \)

does not hold. Secondly: the postulate of parallels does not refer to a finite figure and is therefore not obvious. For these reasons, the postulate of parallels was suspected not
to be an independent axiom and it was tried to deduce it from the other axioms. But all attempts failed. This contradiction could not be resolved within Euclidean geometry but only by a generalization.

This generalization was initiated by Johannes Kepler (1571-1630) when he introduced so-called improper elements:

An improper point (or point at infinity) is the intersection of two straight lines, which are parallel in the Euclidean sense. The set of all improper points is an improper straight line.

Now, with the inclusion of improper elements the symmetry is reconstituted:

Two straight lines intersect in exactly one (possibly improper) point. Two points are connected by exactly one straight line.

This symmetric pair of geometric results is the simplest example of the duality principle in projective geometry of the plane.

**Theorem:** If in a true statement in plane projective geometry the phrase “are connected” is replaced by the phrase “intersect in” and the concept “point” by the concept “straight line”, the resulting statement is also true.

The duality principle is the key to a by-pass technique for the proof of geometrical results:

1. Map a given figure in its dual figure by mapping points into straight lines and straight lines into points
2. Prove a result for the dual figure.
3. Retranslate the obtained result into a result for the original figure using the duality principle

In projective geometry such a mapping, which transforms a figure in its dual figure is called correlation. One simple example for such a correlation is the polarity at the unit circle. Here a point P, the so called pole, is mapped into a straight line p, the so-called polar, in the following way: The tangents t1 and t2 from the pole P to the unit circle are constructed. The line p which passes through the points of contact of the tangents with the unit circle is the polar p. Vice versa a straight line is mapped into a point in the following way: In the points where the straight lines intersects the unit circle the tangents to the unit circle are constructed. The tangents intersect each other in a point.
which is the image of the straight line. Also here in the two mappings the duality principle is visible. For the analytic description of a correlation the use of so-called homogeneous coordinates, introduced by Julius Plücker (1801-1868), is useful. Homogeneous coordinates \((x_1, x_2, x_3)\) of a finite point \((x, y)\) in the plane are any three real numbers, for which holds
\[
\frac{x_1}{x_3} = x, \quad \frac{x_2}{x_3} = y.
\]

Coordinates \((x_1, x_2, 0)\) for which
\[
\frac{x_2}{x_1} = \lambda
\]
describe the point at infinity in direction of the slope \(\lambda\). Using homogeneous coordinates the equation of a straight line \(g\)
\[
a_1x + a_2y + a_3 = 0
\]
can be written as
\[
a_1x_1 + a_2x_2 + a_3x_3 = 0.
\]

For this reason the coefficients \((a_1, a_2, a_3)\) are called homogeneous coordinates of the straight line \(g\).

In homogeneous coordinates, the polarity at the unit circle has the simple form
\[
b_1 = x_1 \quad b_2 = x_2 \quad b_3 = -x_3,
\]
where \((x_1, x_2, x_3)\) are the homogeneous coordinates of the pole \(P\) and \((b_1, b_2, b_3)\) are the homogeneous coordinates of the corresponding polar \(p\). The analytic description of the polarity at the unit circle is rather simple, but not the simplest possible one. An even simpler polarity is obtained, if the unit circle
\[
y_1^2 + y_2^2 - y_3^2 = 0
\]
is replaced by the complex unit circle
\[
y_1^2 + y_2^2 + y_3^2 = 0.
\]

Then the polarity at the complex unit circle is simply
i.e. the homogeneous coordinates of the polar are simply the homogeneous coordinates of the pole.

**Contact-transformations**

We now consider an ordinary differential equation

\[ F(t, x(t), x'(t)) = 0. \]

An ordinary differential equation can be seen as an equation between the solution curve \((t, x(t))\) and the inclinations of its tangents \(x - x_0 = x'(t_0)(t - t_0)\). Sophus Lie (1842-1899) called the collection of a curve (or more general a hypersurface) and its tangents (or more general its tangent spaces) an *elementverein*.

It is natural to describe an *elementverein* in homogeneous coordinates

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
    t \\
    x(t) \\
    1 \\
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
    -x'(t) \\
    1 \\
    x't - x \\
\end{bmatrix}.
\]

Once the step to homogeneous coordinates is done, one could be curious what a correlation would do with the *elementverein*. For a start the simplest possible correlation, the polarity at the complex unit circle, is studied:

\[
\begin{bmatrix}
    y_1 \\
    y_2 \\
    y_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3 \\
\end{bmatrix}
\begin{bmatrix}
    -x' \\
    1 \\
    x't - x \\
\end{bmatrix}.
\]

If Cartesian coordinates are reintroduced

\[
\tau = \frac{y_1}{y_2} = -x', \quad \xi = \frac{y_3}{y_2} = x't - x,
\]
the polarity at the complex unit circle leads to the one-dimensional version of the
Legendre transformation, well known from thermo-dynamics. For the Legendre
transformation the following relation holds

\[ d\xi - \frac{d\xi}{d\tau} d\tau = d(x't - x) - tdx' = dx't + x'dt - dx + tdx' = -(dx - \frac{dx}{dt} dt). \]

This relation is the prototype for the key property of so-called contact-transformations.
A mapping

\[ \tau = \tau(t, x, p), \quad \pi = \pi(t, x, p), \quad \xi = \xi(t, x, p) \]

is called a contact-transformation, if there is a non-vanishing function \( \rho(t, x, p) \) with

\[ d\xi - \pi^T d\tau = \rho(dx - p^T dx). \]

The contact condition guarantees that an element \( x(t) \) with \( p \) as normals to the
tangent spaces is again mapped into another element \( \xi(\tau) \) with the normals \( \pi \). As
a convention, the quantities \( t, \tau \) are called coordinates and adjoint coordinates, the
quantities \( p, \pi \) are called impulses and adjoint impulses and finally the functions \( x, \xi \)
are called potential and adjoint potential, respectively. For a contact-transformation, it
is characteristic that information about coordinates is exchanged with information
about impulses. For this reason, contact-transformations can be used to simplify
differential equations.

**Gravity-space approach**

The following problem is usually called boundary value problem of physical geodesy:
Assume that \( S \subseteq \mathbb{R}^3 \) is a smooth, closed, orientable surface. On this surface two
functions are given:

\[ v : S \to \mathbb{R}^1, \quad g : S \to \mathbb{R}^3. \]

Find the surface \( S \) and a function \( V \), defined in the exterior of \( S \), so that

\[ \Delta V(x) = 0, \quad x \in \text{ext } S \]

\[ V|_S = v, \quad \nabla V|_S = g. \]

Here, \( S \) stands for the surface of the Earth and \( V \) for its gravitational potential. The
boundary value problem of Physical Geodesy is a nonlinear, free boundary value
problem. It is a free boundary value problem because the boundary surface is among
the unknowns of the problem and it is a nonlinear problem because the boundary
conditions couple the two unknowns \( V, S \) in a nonlinear way. Typical boundary value problems have a known boundary surface and linear boundary conditions. Therefore, the question arises if it is possible to transform the free boundary value problem into a boundary value problem with a fixed boundary?

A closer analysis of the boundary value problem shows that we have a lack of geometric information (information about the geometry of the boundary surface) but a surplus of physical information (two boundary conditions instead of the only necessary single boundary condition). Hence, for a transformation into a boundary value problem with fixed boundary one could try to exchange the surplus physical information with the lacking geometric information. An obvious candidate for such an exchange is a contact-transformation. In order to apply a contact-transformation, one has to find an interpretation of the boundary value problem of Physical Geodesy as an elementverein. The quadruple \((x, V(x))\) is a hypersurface in \( R^4 \). Its tangential spaces are given by the equations

\[
V(x) - V(x_0) = \nabla V(x_0)^T (x - x_0).
\]

In the terminology of contact–transformations, the coordinates are the vectors \( x \), the impulses are the gradients \( \nabla V \) and the potential is the unknown function \( V \). A contact transformation now computes adjoint coordinates \( \xi = \zeta(x, \nabla V, V) \), adjoint impulses \( \nabla \psi = \nabla \psi(x, \nabla V, V) \) and an adjoint potential \( \psi = \psi(x, \nabla V, V) \) so that

\[
d\psi - \nabla \psi^T d\xi = \rho(dV - \nabla V^T dx)
\]

holds. The simplest possible contact-transformation is the Legendre transformation

\[
\xi = \nabla V, \quad \nabla \psi = x, \quad \psi = \xi^T x - V,
\]

coming from the polarity at the complex hyper unit-sphere. This Legendre transformation is the famous gravity-space transformation, discovered by Sansò (1978).
The big benefit of the gravity-space transformation is that the image of the unknown boundary surface $S$ is known:

$$S \rightarrow \Sigma = \nabla V|_S = g.$$ 

Therefore, instead of a free boundary value problem for the potential $V$, we now have a boundary value problem for the adjoint potential $\psi$. Only, the differential equation for $\psi$ and the boundary condition still have to be found. The differential equation can be found because the transformation into the gravity-space is bijective. Hence, the Jacobian

$$\frac{\partial \xi}{\partial x} = \begin{pmatrix} \frac{\partial \xi_i}{\partial x_j} \\ \frac{\partial \xi_j}{\partial x_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_i \partial x_j} \end{pmatrix}$$

is regular. For this reason

$$\begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \\ \frac{\partial \psi}{\partial \xi_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial \psi}{\partial x_i} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_i \partial x_j} \end{pmatrix}^{-1}$$

is valid. Since $V$ is harmonic, the following relation holds:

$$0 = Tr \begin{pmatrix} \frac{\partial^2 V}{\partial x_i \partial x_j} \end{pmatrix} = Tr \begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \end{pmatrix}^{-1} = \frac{-1}{\det \begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \end{pmatrix}} \left[ Tr \begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \end{pmatrix}^2 - \left[ Tr \begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \end{pmatrix} \right]^2 \right].$$

This is a nonlinear partial differential equation for the adjoint potential $\psi$. The boundary condition simply follows from the definition of the adjoint potential

$$(\xi^T \nabla \psi - \psi)|_{\Sigma} = \nu.$$ 

In this way, the gravity-space transformation indeed changed the free boundary value problem into a boundary value problem with a fixed boundary, which is much simpler to treat mathematically.

The only drawback of the gravity-space technique is the asymptotic behavior of the adjoint potential:

$$\psi(\xi) = O(\|\xi\|^{-1}), \quad \xi \rightarrow 0,$$

which means that $\psi$ is not differentiable in the origin and therefore cannot fulfill the differential equation.
Regular gravity-space approach

To find a gravity-space transformation which still transforms a free into a fixed boundary value problem but does not suffer from the singularity in the origin, one has to find the reasons behind the singularity of Sansò’s gravity-space approach. One immediately can see that in this transformation the point at infinity is mapped onto the origin. Because the concept of differentiability looses its meaning in the point at infinity, this lack of differentiability is transported into the origin. For a regular gravity-space transformation a contact-transformation has to be found, which has the point at infinity as fixed point. Such a regular gravity-space transformation was proposed by Keller (1987)

\[ \xi = -\sqrt{GM} \frac{\nabla V}{\|\nabla V\|^2}, \quad \nabla \psi = \left( \sum_j \alpha_{i,j} x_j \right), \quad \psi = \nabla V^T x - V, \quad \alpha_{i,j} = - \frac{GM}{\|\xi\|^2} \left( \delta_{ij} - \frac{\xi_i \xi_j}{\|\xi\|^2} \right). \]

This transformation is indeed a contact-transformation, because of

\[
d\psi - d\xi^T \nabla \psi = d(\nabla V^T x - V) - d\xi^T \nabla \psi = x^T d\nabla V + \nabla V^T dx - dV - \left( \frac{\partial \xi}{\partial \nabla V} \right)^T (\alpha_{i,j})x = x^T d\nabla V + \nabla V^T dx - dV - \left( (\alpha_{i,j})^{-1} d\nabla V \right)^\top (\alpha_{i,j})x = x^T d\nabla V + \nabla V^T dx - dV - d\nabla V^T x = -(dV - dx^T \nabla V). \]

Furthermore, one can see that the point at infinity is indeed a fixed point of this regular gravity-space transformation. For the asymptotic behavior of the adjoint potential \( \psi \) one obtains:
\[
\psi(\xi) = O\left(\frac{1}{|\xi|}\right), \quad \xi \to \infty,
\]
and therefore the singularity has disappeared. The boundary value problem for the adjoint potential \(\psi\), is similar to the differential equation in the classical gravity-space approach given by

\[
0 = \frac{1}{\det \Psi} \left(Tr \Psi^2 - \{Tr \Psi\}^2\right) - \left(\frac{1}{2} \xi^T \nabla \psi + \psi\right)_{|\xi|} = \nu,
\]

only that the matrix \(\Psi\) is now the following linear combination of the Hessian and the Jacobian of the adjoint potential \(\psi\)

\[
\Psi = (\psi_{i,j}) = \left(\gamma_{i,j,k,l} \frac{\partial^2 \psi}{\partial \xi_k \partial \xi_l} - \sum_m \beta_{i,m,j} \frac{\partial \psi}{\partial \xi_m}\right),
\]

and not the Hessian alone. Interesting is also the image of the Earth’s surface under the regular gravity-space transformation. If we denote the spherical normal potential by \(V_0\)

\[
V_0(x) = \frac{GM}{|x|},
\]

then the following is true:

\[
\nabla V_0(\xi) = -GM \frac{\nabla V(x)}{\|\nabla V(x)\|^{3/2}} = \frac{\nabla V(x) \frac{GM^{3/2}}{\|\nabla V(x)\|^{1/2}}}{\left(\sqrt{GM} \frac{\nabla V(x)}{\|\nabla V(x)\|^{3/2}}\right)^3} = \nabla V(x).
\]

This means the regular gravity-space transformation maps the Earth’s onto the gravimetric telluroid. In particular: If the gravitational potential \(V\) was spherical, the regular gravity-space transformation would be the identical transformation.

**Summary**

It was shown that the transformation of the free boundary value problem of physical eodesy into a boundary value problem with a fixed boundary is essentially an exchange of the lacking geometrical information about the boundary with a surplus of physical information given at the boundary. This exchange is possible by not considering geometric and physical information as separate entities but by fusion them into an
elementverein. The exchange itself is carried out by a contact-transformation which maps one elementverein into another. The simplest possible contact-transformation, the Legendre transformation, leads to the classical gravity-space approach, which transforms the free boundary value problem of physical geodesy into a boundary value problem with a fixed boundary but suffers from a singularity in the origin. The fact that the classical gravity-space transformation maps the point at infinity into the origin was identified as the cause of this singularity. As an alternative a more complicated, regular gravity-space transformation was proposed, which leads to a regular boundary value problem with a fixed boundary.

The paper covers only the theoretical and not the numerical aspects of the gravity-space technique. Numerical questions are discussed in Austen (2008) and Austen, Keller (2008).

References

