A closed-form solution of the nonlinear pseudo-ranging equations (GPS)

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Summary

For the closed-form solution of the nonlinear pseudo-ranging equations as they appear in satellite positioning ("GPS": "Global Problem Solver") we present two three-step algorithms. In case of an unknown ground receiver position and an unknown stationary receiver time bias the solution point is the intersection of four three-dimensional spherical cones (three-dimensional algebraic manifold). The first closed-form solution algorithm is based upon a geocentric frame of reference: A three-by-three matrix has to be inverted. The characteristic inverse contains the volume element as well as the area elements of the four satellite points tetrahedron as structural elements. The solution bifurcates; an admissible solution has to be tested against prior information. In contrast, the second closed form solution algorithm is based upon a Moebius barycentric frame of reference: A three-by-three matrix has to be inverted. The characteristic inverse contains the distances and the face angles of the four satellite points tetrahedron as structural elements. Numerical examples are presented to demonstrate our algorithms.
0 Introduction

Two types of algorithms for a closed-form solution of the pseudo-ranging equations of nonlinear type as they appear as first order models in satellite positioning ("GPS": "Global Problem Solver") are presented. In paragraph one we set up the two-way classification model for pseudo-ranging observations between a satellite transmitter and a ground receiver which includes the stationary receiver time bias, the stationary transmitter bias and the stationary mixed receiver-transmitter bias. As long as we pay attention to the unknown stationary receiver time bias only, we prove that the unknown receiver point on the ground is the intersection point of four three-dimensional spherical cones embedded into $\mathbb{R}^4$. In a geometric frame of references those four quadratic equations with four unknowns are solved by a three-step reduction algorithm which is outlined. The solution space is non-unique; two solutions are produced which have to be tested with respect to admissibility from prior information. A special highlight is the definition of the bifurcation problem. Within the closed form algorithm a three-by-three matrix has to be inverted, its inverse contains the volume element and the area elements of faces of the tetrahedron (four-dimensional simplex) built on the four given satellite transmitter points. In contrast, in paragraph two we aim at a closed-form solution of the four nonlinear pseudo-ranging equations in a Moebius barycentric frame of reference, namely an affine basis constituted by relative placement vectors of the satellite points tetrahedron (four-dimensional simplex). The algorithm focusses again on the inversion of a three-by-three matrix; the inverse contains the distances between satellite transmitter points and the scalar products of relative tetrahedron placement vector or angles of sides of the faces of the tetrahedron.

The closed-form solution of the nonlinear pseudo-ranging equations in terms of the two algorithms contains all the structural elements of "pseudo 4P" as being discussed by A. Kleusberg (1994) and H. Lichtenegger (1995) to which we refer. The critical configurations analyzed by T. A. Wunderlich (1993) appear as singular volume and/or area elements. The bifurcation problem has been already identified by J. S. Abel and J. W. Chaffee (1991), S. Bancroft (1985) and J. Chaffee and J. Abel (1994). In the context of geodetic positioning, positioning in photogrammetry, computer and machine vision as well as robotics Moebius barycentric coordinates have been pioneered by W. Pachelski (1994), E. Grafarend and J. Shan (1996 a, b).
1 Closed-form solution of the nonlinear pseudo-ranging four point problem \((\textit{pseudo 4P})\) in geocentric coordinates

At first let us set up the nonlinear observational equation for pseudo-ranging between a satellite transmitter \(P^j \sim (x^j, y^j, z^j)\) and a ground receiver \(P \sim (x, y, z)\) with respect to Cartesian coordinates \(\{e_1, e_2, e_3; o\}\). Actually a ground point on the Earth surface receives at ground proper time a satellite transmitter signal at satellite proper time. The proper times are transformed, as outlined by \(V.\text{ Schwarze} (1996)\) for instance, into coordinate time \(T_i\) of type "ground" and \(T^i\) of type "satellite". We apply the following notation for indices: superscripts \(j \in \{0, 1, 2, 3\}\), subscripts \(i \in \mathbb{N}\). As soon as we multiply the coordinate time delay \(T_i - T^j\) by the vacuum velocity \(c\) of the electromagnetic signal - refraction-dispersion reduction has been performed in the peripheral mode - we gain the representation \((1.1)\) of \(Box 1.1\), namely the pseudo-range which is biased by a stationary receiver time offset \(\alpha_i\), a stationary transmitter time offset \(\beta^j\) and a stationary mixed receiver-transmitter range bias \(\gamma^j_i\). \((1.1iii)\) constitutes a two-way classification model. An empirical analysis of GPS observations by \(W.\text{ Lindlohr} (1989)\) has documented that the stationary receiver time bias \(\alpha_i\) is orders of magnitude larger than the other bias terms \(\beta^j\) and \(\gamma^j_i\). Accordingly we have neglected in the first order model of the observational equation \((1.1iii)\) of pseudo-ranging all bias terms but the stationary receiver time bias \(\alpha_i\). In addition, we have represented the product of vacuum velocity \(c\) of the electromagnetic signal and the coordinate time delay \(t_i - t^j\) by the three-dimensional Euclidean distance. In summary, the observational equation \((1.1iii)\) of pseudo-ranging type contains four unknowns, namely the Cartesian coordinates \(x_1 = x, x_2 = y, x_3 = z\) of the unknown ground point \(P\), namely in an orthonormal geocentric frame of reference \(\{e_1, e_2, e_3; o\}\), e.g. WGS 84, and the stationary receiver range bias \(x_4 = c\alpha\) if the Cartesian coordinates \((x^j, y^j, z^j)\) of satellite points \(P^j\) are given. The pseudo-ranging four point problem \("pseudo 4P\") is defined as the problem to determine from four observed pseudo-ranges \(y(P, P^j)\) to four satellite transmitter of given geocentric position the four unknowns \((x_1, x_2, x_3, x_4)\) of type position \((x_1, x_2, x_3)\) and stationary receiver range bias \(x_4\). In the following we deal with the given quantities of type observations \(d_j := y(P, P^j)\) and coordinates \((a_j, b_j, c_j) := (x^i, y^j, z^j)\) as parameters which determine the algebraic manifold \((1.3)\), namely the three-dimensional spherical cone \(C^3\). In geometric terms \"pseudo 4P\" can be described as following: The unknown point \(P \sim (x, y, z)\) within \"pseudo 4P\" is determined by the intersection of four three-dimensional spherical cones (three-dimensional algebraic manifold), namely \(P \in C^3 \cup C^3_1 \cup C^3_2 \cup C^3_3\).
Secondly we shall demonstrate a reduction algorithm to determine the unknown vector \([x_1, x_2, x_3, x_4]\) from the four polynomial equations (1.6) of Box 1.2, in particular which solves "pseudo 4P". The first algorithmic step is generated by the difference operation, namely equations (1.6ii), (1.6iii) and (1.6iv) minus (1.6i). The quadratic terms cancel within (1.7), (1.8) and we are left with the linear system of equations (1.9), (1.10), (1.11). Inversion by means of (1.12), (1.13), (1.14), (1.15), (1.16), (1.17) leads to the representation of \([x_1, x_2, x_3] (x_4)\), namely a linear function \((x_1, x_2, x_3)\) of \(x_4\), (1.18)-(1.21). The involved inverse matrix is geometrically interpreted by means of the volume vol \(\{x^0, x^1, x^2, x^3\}\) of the four-dimensional simplex, the tetrahedron built up by four satellite points \(P^j\) as well as area elements area \(\{x^0, x^1, x^2\}\), area \(\{x^0, x^2, x^3\}\), area \(\{x^0, x^1, x^3\}\) of its faces. The second algorithmic step presented in Box 1.3 is generated by replacing successively \(x_1(x_4), x_2(x_4), x_3(x_4)\) in the first equation (1.6i). In this way we are led by means of (1.23), (1.24), (1.25) to the quadratic equation (1.26) for \(x_4\), in particular with the two solutions \(x^+_4, x^-_4\). Box 1.4 reviews after the first forward reduction steps the third algorithmic step of backward type. As soon as we have computed the stationary receiver range bias \(x^+_4\) from (1.26) we resubstitute \(x^+_4\) in (1.19i), (1.19ii), (1.19iii) in order to compute \(x^+_1, x^+_2, x^+_3\). From prior information about the location of the receiver point \(P \sim (x_1, x_2, x_3)\) we have to take reference in order to decide whether \((x^+_1, x^+_2, x^+_3)\) or \((x^-_1, x^-_2, x^-_3)\) is admissible. Of particular importance is the bifurcation problem, namely the problem to identify the point where the solution of the quadratic equation (1.26) bifurcates. There is only one "pseudo 4P" solution if within (1.27) the square root vanishes or \(4AC = B^2\). A more detailed analysis of the bifurcation manifold is given elsewhere.
Box 1.1: The observational equation for pseudo-ranging, two-way classification model

\[ i, j, \in \mathbb{N} \]

\[ y(P_i, P^j) = c(T_i - T^j) = c(t_i + \alpha_i - t^j - \beta^j) + \gamma_i^j \] (1.1i)

\[ y(P_i, P^j) = c(t_i - t^j) + c\alpha_i - c\beta^j + \gamma_i^j \] (1.1ii)

"set of stationary bias parameters"

\[ \{\alpha, \beta, \gamma\} \in \mathbb{R} \]

\( \alpha_i \) stationary receiver time bias (real number)

\( c\alpha_i \) stationary receiver range bias (real number)

\( \beta^j \) stationary transmitter bias (real number)

\( c\beta^j \) stationary transmitter range bias (real number)

\( \gamma_i \) stationary mixed receiver-transmitter range bias (real number)

"if the stationary transmitter bias \( \beta^j = 0 \) as well as the stationary mixed receiver-transmitter bias \( \gamma_i^j = 0 \) is set to zero, the pseudo-ranging observational equation reads as following:"

\[ y(P, P^j) = c(t - t^j) + c\alpha = \sqrt{(x - x^j)^2 + (y - y^j)^2 + (z - z^j)^2 + c\alpha} \] (1.1iii)

\[ \begin{align*}
  x_1 & := x \\
  x_2 & := y \\
  x_3 & := z \\
  x_4 & := c\alpha
\end{align*} \] (1.2i)

\[ \begin{align*}
  a_j & := x^j \\
  b_j & := y^j \\
  c_j & := z^j \\
  d_j & := y(P, P^j)
\end{align*} \] (1.2ii)

\[ [y(P, P^j) - c\alpha]^2 = (x - x^j)^2 + (y - y^j)^2 + (z - z^j)^2 \quad \forall j \in \{0, 1, 2, 3\} \] (1.3)

\[ (x_1 - a_j)^2 + (x_2 - b_j)^2 + (x_3 - c_j)^2 - (x_4 - d_j)^2 = 0 \quad \forall j \in \{0, 1, 2, 3\} \] (1.4)

"three-dimensional spherical cone"

\[ C^3 := \{x \in \mathbb{R}^4|(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - (x_4 - d)^2 = 0, a, b, c, d \in \mathbb{R}\} \] (1.5)
Box 1.2: Determination of the unknown point from the pseudo-ranging four point problem (*pseudo 4P*), intersection of four spherical cones, the first reduction step

\[
\begin{align*}
(x_1 - a_0)^2 + (x_2 - b_0)^2 + (x_3 - c_0)^2 - (x_4 - d_0)^2 &= 0 \\
(x_1 - a_1)^2 + (x_2 - b_1)^2 + (x_3 - c_1)^2 - (x_4 - d_1)^2 &= 0 \\
(x_1 - a_2)^2 + (x_2 - b_2)^2 + (x_3 - c_2)^2 - (x_4 - d_2)^2 &= 0 \\
(x_1 - a_3)^2 + (x_2 - b_3)^2 + (x_3 - c_3)^2 - (x_4 - d_3)^2 &= 0
\end{align*}
\]  

(1.6)

subject to

\[
\begin{align*}
P^0 &\sim (x^0, y^0, z^0) \sim (a_0, b_0, c_0) \\
P^1 &\sim (x^1, y^1, z^1) \sim (a_1, b_1, c_1) \\
P^2 &\sim (x^2, y^2, z^2) \sim (a_2, b_2, c_2) \\
P^3 &\sim (x^3, y^3, z^3) \sim (a_3, b_3, c_3)
\end{align*}
\]

\[y(P, P^0) = d_0, \ y(P, P^1) = d_1, \ y(P, P^2) = d_2, \ y(P, P^3) = d_3\]

"parameters \((a_0, \ldots, d_3)\) of the spherical cones, \(P \in \mathbb{C}_0^3 \cup \mathbb{C}_1^3 \cup \mathbb{C}_2^3 \cup \mathbb{C}_3^3\)"

**Step one**

"The difference operation"

\(j, k \in \{0, 1, 2, 3\}\)

\[
\begin{align*}
(a_j^2 - a_k^2) - 2(a_j - a_k)x_1 + (b_j^2 - b_k^2) - 2(b_j - b_k)x_2 + (c_j^2 - c_k^2) - 2(c_j - c_k)x_3 - \\
- d_j^2 + d_k^2 + 2(d_j - d_k)x_4 &= 0
\end{align*}
\]  

(1.7)

\[
\begin{align*}
(a_j - a_k)x_1 + (b_j - b_k)x_2 + (c_j - c_k)x_3 - (d_j - d_k)x_4 = \\
= \frac{1}{2}(a_j^2 + b_j^2 + c_j^2 - d_j^2) - \frac{1}{2}(a_k^2 + b_k^2 + c_k^2 - d_k^2) =: e_{jk}
\end{align*}
\]  

(1.8)
\[
(a_1 - a_0)x_1 + (b_1 - b_0)x_2 + (c_1 - c_0)x_3 = (d_1 - d_0)x_4 + e_{10}
\]
\[
(a_2 - a_0)x_1 + (b_2 - b_0)x_2 + (c_2 - c_0)x_3 = (d_2 - d_0)x_4 + e_{20}
\]
\[
(a_3 - a_0)x_1 + (b_3 - b_0)x_2 + (c_3 - c_0)x_3 = (d_3 - d_0)x_4 + e_{30}
\]
\[
e_{10} := \frac{1}{2}[(a_1^2 + b_1^2 + c_1^2 - d_1^2) - (a_0^2 + b_0^2 + c_0^2 - d_0^2)]
\]
\[
e_{20} := \frac{1}{2}[(a_2^2 + b_2^2 + c_2^2 - d_2^2) - (a_0^2 + b_0^2 + c_0^2 - d_0^2)]
\]
\[
e_{30} := \frac{1}{2}[(a_3^2 + b_3^2 + c_3^2 - d_3^2) - (a_0^2 + b_0^2 + c_0^2 - d_0^2)]
\]

\[
A := \begin{bmatrix}
  a_1 - a_0 & b_1 - b_0 & c_1 - c_0 \\
  a_2 - a_0 & b_2 - b_0 & c_2 - c_0 \\
  a_3 - a_0 & b_3 - b_0 & c_3 - c_0 
\end{bmatrix}
= \begin{bmatrix}
  x^1 - x^0 & y^1 - y^0 & z^1 - z^0 \\
  x^2 - x^0 & y^2 - y^0 & z^2 - z^0 \\
  x^3 - x^0 & y^3 - y^0 & z^3 - z^0 
\end{bmatrix}
\]

\[
B := A^{-1} = \frac{\text{adj } A}{\text{det } A}
\]

\[
\text{det } A = \begin{vmatrix}
  a_1 - a_0 & b_1 - b_0 & c_1 - c_0 \\
  a_2 - a_0 & b_2 - b_0 & c_2 - c_0 \\
  a_3 - a_0 & b_3 - b_0 & c_3 - c_0 
\end{vmatrix} = (-1)
\]

\[
-\text{det } A = \begin{vmatrix}
  x^0 & y^0 & z^0 & 1 \\
  x^1 & y^1 & z^1 & 1 \\
  x^2 & y^2 & z^2 & 1 \\
  x^3 & y^3 & z^3 & 1 
\end{vmatrix} = 6 \text{ vol } \{x^0, x^1, x^2, x^3\}
\]

\[
\text{adj } A = 
\begin{bmatrix}
  b_2 - b_0 & c_2 - c_0 & b_3 - b_0 & b_1 - b_0 & b_1 - b_0 & c_1 - c_0 \\
  b_3 - b_0 & c_3 - c_0 & c_3 - c_0 & c_1 - c_0 & b_2 - b_0 & c_2 - c_0 \\
  c_2 - c_0 & a_2 - a_0 & a_1 - a_0 & c_1 - c_0 & c_1 - c_0 & a_1 - a_0 \\
  c_3 - c_0 & a_3 - a_0 & a_3 - a_0 & c_2 - c_0 & c_2 - c_0 & a_2 - a_0 \\
  a_2 - a_0 & b_2 - b_0 & b_1 - b_0 & a_1 - a_0 & a_1 - a_0 & b_1 - b_0 \\
  a_3 - a_0 & b_3 - b_0 & b_3 - b_0 & a_3 - a_0 & a_2 - a_0 & b_2 - b_0 
\end{bmatrix}
\]
adj $A = \begin{bmatrix}
    y^2 - y^0 & z^2 - z^0 & z^2 - z^0 \\
    y^1 - y^0 & y^1 - y^0 & y^1 - y^0 \\
    z^1 - z^0 & z^1 - z^0 & z^1 - z^0 \\
    z^0 - z^0 & z^0 - z^0 & z^0 - z^0 \\
    y^0 - y^0 & y^0 - y^0 & y^0 - y^0 \\
\end{bmatrix}$ \hfill (1.16)

\begin{align*}
\text{area} \{z^0, z^1, x^3\} &= \frac{1}{2} \sqrt{y^0 - y^0 \ z^2 - z^0 \ z^2 - z^0 + z^2 - z^0 \ z^2 - z^0 \ z^2 - z^0 + z^2 - z^0 \ y^0 - y^0 \ y^0 - y^0} \hfill (1.17i) \\
\text{area} \{z^0, z^1, x^3\} &= \frac{1}{2} \sqrt{y^1 - y^0 \ z^1 - z^0 \ z^1 - z^0 + z^1 - z^0 \ z^1 - z^0 \ z^1 - z^0 + z^1 - z^0 \ y^1 - y^0 \ y^1 - y^0} \hfill (1.17ii) \\
\text{area} \{z^0, x^1, x^2\} &= \frac{1}{2} \sqrt{z^0 - z^0 \ z^1 - z^0 \ z^1 - z^0 + z^1 - z^0 \ z^1 - z^0 \ z^1 - z^0 + z^1 - z^0 \ y^1 - y^0 \ y^1 - y^0} \hfill (1.17iii) \\
\end{align*}

\begin{equation}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix} = B \begin{bmatrix}
    d_1 - d_0 \\
    d_2 - d_0 \\
    d_3 - d_0 \\
\end{bmatrix} x_4 + B \begin{bmatrix}
    c_{10} \\
    c_{20} \\
    c_{30} \\
\end{bmatrix} \hfill (1.18)
\end{equation}

\begin{align*}
x_1 &= (b_{11}(d_1 - d_0) + b_{12}(d_2 - d_0) + b_{13}(d_3 - d_0))x_4 + b_{11} \epsilon_{10} + b_{12} \epsilon_{20} + b_{13} \epsilon_{30} =: c_{14} x_4 + c_{15} \hfill (1.19i) \\
x_2 &= (b_{21}(d_1 - d_0) + b_{22}(d_2 - d_0) + b_{23}(d_3 - d_0))x_4 + b_{21} \epsilon_{10} + b_{22} \epsilon_{20} + b_{23} \epsilon_{30} =: c_{24} x_4 + c_{25} \hfill (1.19ii) \\
x_3 &= (b_{31}(d_1 - d_0) + b_{32}(d_2 - d_0) + b_{33}(d_3 - d_0))x_4 + b_{31} \epsilon_{10} + b_{32} \epsilon_{20} + b_{33} \epsilon_{30} =: c_{34} x_4 + c_{35} \hfill (1.19iii) \\
\end{align*}

subject to
\begin{align*}
c_{14} &:= b_{11}(d_1 - d_0) + b_{12}(d_2 - d_0) + b_{13}(d_3 - d_0) \hfill (1.20i) \\
c_{24} &:= b_{21}(d_1 - d_0) + b_{22}(d_2 - d_0) + b_{23}(d_3 - d_0) \hfill (1.20ii) \\
c_{34} &:= b_{31}(d_1 - d_0) + b_{32}(d_2 - d_0) + b_{33}(d_3 - d_0) \hfill (1.20iii) \\
c_{15} &:= b_{11} \epsilon_{10} + b_{12} \epsilon_{20} + b_{13} \epsilon_{30} \hfill (1.21i) \\
c_{25} &:= b_{21} \epsilon_{10} + b_{22} \epsilon_{20} + b_{23} \epsilon_{30} \hfill (1.21ii) \\
c_{35} &:= b_{31} \epsilon_{10} + b_{32} \epsilon_{20} + b_{33} \epsilon_{30} \hfill (1.21iii) \\
\end{align*}
Box 1.3: Determination of the unknown point from the pseudo-ranging four points problem (pseudo 4P), intersection of four spherical cones, the second reduction step

**Step two**

\[ C_0^3 := \{ x \in \mathbb{R}^4 | (x_1-a_0)^2 + (x_2-b_0)^2 + (x_3-c_0)^2 - (x_4-d_0)^2 = 0, a, b, c, d \in \mathbb{R} \} \]  

"(1.19i), (1.19ii), (1.19iii) into (1.22)"

\[ (c_{14}x_4 + c_{15} - a_0)^2 + (c_{24}x_4 + c_{25} - b_0)^2 + (c_{34}x_4 + c_{35} - c_0)^2 - (x_4 - d_0)^2 = 0 \]  

\[ (c_{14}^2 + c_{24}^2 + c_{34}^2 - 1)x_4^2 - 2[(a_0 - c_{15})c_{14} + (a_0 - c_{25})c_{24} + (a_0 - c_{35})c_{34} - d_0]x_4 + \]  

\[ +[(a_0 - c_{15})^2 + (b_0 - c_{25})^2 + (c_0 - c_{35})^2 - d_0^2] = 0 \]  

\[ A := c_{14}^2 + c_{24}^2 + c_{34}^2 - 1 \]  

\[ B := -2[(a_0 - c_{15})c_{14} + (a_0 - c_{25})c_{24} + (a_0 - c_{25})c_{34} - d_0] \]  

\[ C := (a_0 - c_{15})^2 + (b_0 - c_{25})^2 + (c_0 - c_{35})^2 - d_0^2 \]  

\[ Ax_4^2 + Bx_4 + C = 0 \]  

\[ x_4^\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]
Box 1.4: Determination of the unknown point from the pseudo-ranging four point problem \((pseudo\ 4P)\), intersection of four spherical cones, the third reduction step

**Step three**

"Insert (1.27) \(x_4^\pm\) into (1.19i), (1.19ii), (1.19iii) in order to compute \(x_1^\pm, x_2^\pm, x_3^\pm\)"

"Decide from prior information upon the admissible solution \((x_1^+, x_2^+, x_3^+, x_4^+)\) or \((x_1^-, x_2^-, x_3^-, x_4^-)\)"

"Bifurcation of pseudo 4P"

"The quadratic equation (1.26) has only one solution at the bifurcation point":

\[
AC = B^2
\]  

(1.28)

\[
(c_{14}^2 + c_{24}^2 + c_{34}^2 - 1)[(a_0 - c_{15})^2 + (b_0 - c_{25})^2 + (c_0 - c_{35})^2 - d_0^2] =

= [(a_0 - c_{15})c_{14} + (a_0 - c_{25})c_{24} + (a_0 - c_{35})c_{34} - d_0]^2
\]  

(1.29)

To demonstrate this algorithm we follow the four GPS satellites configuration presented by Kleusberg (1994). The satellite positions and pseudo-ranging observations are listed in Table 1. Table 2 shows the computational results.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(x^i)</th>
<th>(y^i)</th>
<th>(z^i)</th>
<th>(d_i)</th>
</tr>
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<tr>
<td>0</td>
<td>14832308.660</td>
<td>-20466715.890</td>
<td>-7428634.750</td>
<td>24310764.064</td>
</tr>
<tr>
<td>1</td>
<td>-15799854.050</td>
<td>-13301129.170</td>
<td>17133838.240</td>
<td>22914600.784</td>
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<td>-11867672.960</td>
<td>23716920.130</td>
<td>20628809.405</td>
</tr>
<tr>
<td>3</td>
<td>-12480273.190</td>
<td>-23382560.530</td>
<td>3278472.680</td>
<td>23422377.972</td>
</tr>
</tbody>
</table>
Table 2: Solution of “pseudo 4P” in geocentric coordinates

Two possible biases \((x_4)\):

\[
x^+_4 = -100.0006 \quad x^-_4 = -57479918.164
\]

Two possible user receiver positions:

\[
x^+_1 = 1111590.460 \quad x^+_2 = -4348258.631 \quad x^+_3 = 4527351.820
\]
\[
x^-_1 = -2892123.412 \quad x^-_2 = 7568784.349 \quad x^-_3 = -7209505.102
\]

The admissible solution of user receiver position:

\[
x^+_1 = 1111590.460 \quad x^+_2 = -4348258.631 \quad x^+_3 = 4527351.820
\]

2 Closed-form solution of the nonlinear pseudo-ranging four point problem (pseudo 4P) in barycentric coordinates

At first in order to generate Moebius barycentric coordinates of the points \(P\) and \(P^i\), respectively, we introduce the affine basis

\[
\{ P^0 P^1, P^0 P^2, P^0 P^3 \} \sim \{ x^1 - x^0, x^2 - x^0, x^3 - x^0 \}
\]

which span an \(\mathbb{R}^3\) equipped with a general metric \(g_{\mu\nu}\). With respect to the tetrahedron \(\{ P^0, P^1, P^3 \}\), Figure 2 is a visualization of the affine basis subject to affine geometry. Note that an affine basis is defined as a basis of an \(\mathbb{R}^3\) which is translational invariant or equivariant under the action of the translation group. Relative to \(P^0 \sim x^0\) the point \(P \sim x\) can be represented in the affine basis by \((2.1), (2.2), (2.3)\) of Box 2.1 where \(\{\lambda_1, \lambda_2, \lambda_3\}\) are the Moebius barycentric coordinates of \(P - P^0 \sim x - x^0\).

Box 2.1: Moebius barycentric coordinates, affine basis

\[
x - x^0 = (x^1 - x^0)\lambda_1 + (x^2 - x^0)\lambda_2 + (x^3 - x^0)\lambda_3
\]

\[
\begin{bmatrix}
x - x^0 \\
y - y^0 \\
z - z^0
\end{bmatrix} =
\begin{bmatrix}
x^1 - x^0 & x^2 - x^0 & x^3 - x^0 \\
y^1 - y^0 & y^2 - y^0 & y^3 - y^0 \\
z^1 - z^0 & z^2 - z^0 & z^3 - z^0
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} = A^T
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} = (A^{-1})^T
\begin{bmatrix}
x - x^0 \\
y - y^0 \\
z - z^0
\end{bmatrix}
\]

"see (1.12)-(1.17) for the representation of \(A^{-1}\)"
Moebius barycentric coordinates are implemented in the reduction algorithm at the level of (1.9), (1.10) and (1.11) as outlined in Box 2.2. By means of \([x_1, x_2, x_3]^T = A^T[\lambda_1, \lambda_2, \lambda_3]^T\) the geocentric coordinates \((x_1, x_2, x_3)\) within the system of linear equations (2.4) are replaced by Moebius barycentric coordinates \((\lambda_1, \lambda_2, \lambda_3)\). By continuing the second algorithmic step of forward type - now in terms of Moebius barycentric coordinates - we are led to the system of linear equations (2.6), (2.7). The solution \(x_4^\pm\) of (1.27) within (2.6) leads to the Moebius barycentric coordinates (2.7) where the matrix \(A A^T\) has to be inverted. While the inverse \(A^{-1}\) by means of (1.13)-(1.17) has been built up on volume and area elements of the tetrahedron \(\{P^0, P^1, P^2, P^3\}\), the inverse \((A A^T)^{-1}\) is based upon products of distances of type \(\|x^j - x^0\|^2\) as well as scalar products of type \(\langle x^j - x^0 | x^i - x^0 \rangle = \|x^j - x^0\|\|x^i - x^0\| \cos \psi_{ij}\) as being illustrated by (2.9), (2.10), (2.11). The final algorithmic step is generated by the transformation of Moebius barycentric coordinates \((\lambda_1, \lambda_2, \lambda_3)\) into geocentric coordinates \((x_1, x_2, x_3)\) by means of (2.12).
Box 2.2: Determination of the unknown point from the pseudo-ranging four points problem (pseudo 4P), intersection of four spherical cones, barycentric coordinates

"(1.19)-(1.11), (2.2)"

\[
[x_1, x_2, x_3] = [x, y, z]
\]

\[
A \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  d_1 - d_0 \\
  d_2 - d_0 \\
  d_3 - d_0
\end{bmatrix} x_4 + \begin{bmatrix}
  \epsilon_{10} \\
  \epsilon_{20} \\
  \epsilon_{30}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  x^0 \\
  y^0 \\
  z^0
\end{bmatrix} + A^T \begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \lambda_3
\end{bmatrix}
\]

\[
A A^T \begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \lambda_3
\end{bmatrix} = \begin{bmatrix}
  d_1 - d_0 \\
  d_2 - d_0 \\
  d_3 - d_0
\end{bmatrix} x_4 + \begin{bmatrix}
  \epsilon_{10} \\
  \epsilon_{20} \\
  \epsilon_{30}
\end{bmatrix} - A \begin{bmatrix}
  y^0 \\
  z^0
\end{bmatrix}
\]

"(2.3)\Rightarrow(2.7)"

\[
\begin{bmatrix}
  \lambda^+ \\
  \lambda^{+2} \\
  \lambda^{+3}
\end{bmatrix} = (AA^T)^{-1} \begin{bmatrix}
  d_1 - d_0 \\
  d_2 - d_0 \\
  d_3 - d_0
\end{bmatrix} x^+ + \begin{bmatrix}
  \epsilon_{10} \\
  \epsilon_{20} \\
  \epsilon_{30}
\end{bmatrix} - A \begin{bmatrix}
  y^0 \\
  z^0
\end{bmatrix}
\]

\[
AA^T = \begin{bmatrix}
  \|x^1 - x^0\|^2 & <x^1 - x^0|z^1 - z^0> & <x^1 - x^0|z^1 - z^0> \\
  <x^2 - x^0|z^1 - z^0> & \|x^2 - x^0\|^2 & <x^2 - x^0|z^2 - z^0> \\
  <x^3 - x^0|z^1 - z^0> & <x^2 - x^0|z^3 - z^0> & \|x^3 - z^0\|^2
\end{bmatrix}
\]

\[
(AA^T)^{-1} = \frac{\text{adj} AA^T}{\det AA^T} = \frac{\text{adj} AA^T}{(\det A)^2}
\]

\[
\det A = 36 \text{vol}^2 \{x^0, x^1, x^2, x^3\}
\]

adj \quad AA^T :=

\[
\frac{\|x^1 - x^0\|^2\|x^2 - x^0\|^2\|x^3 - x^0\|^2}{\|x^1 - x^0\|^2\|x^2 - x^0\|^2\|x^3 - x^0\|^2} - \frac{<x^1 - x^0|x^1 - x^0>\|x^2 - x^0\|^2\|x^3 - x^0\|^2}{<x^1 - x^0|x^1 - x^0>\|x^2 - x^0\|^2\|x^3 - x^0\|^2} - \frac{<x^1 - x^0|x^2 - x^0>\|x^3 - x^0\|^2}{<x^1 - x^0|x^2 - x^0>\|x^3 - x^0\|^2}
\]

\[
\begin{bmatrix}
  x^+ \\
  x^{+2} \\
  x^{+3}
\end{bmatrix} = \begin{bmatrix}
  x^0 \\
  y^0 \\
  z^0
\end{bmatrix} + A^T(AA^T)^{-1} \begin{bmatrix}
  d_1 - d_0 \\
  d_2 - d_0 \\
  d_3 - d_0
\end{bmatrix} x^+ + \begin{bmatrix}
  \epsilon_{10} \\
  \epsilon_{20} \\
  \epsilon_{30}
\end{bmatrix} - A \begin{bmatrix}
  y^0 \\
  z^0
\end{bmatrix}
\]

\[
(2.11)
\]

\[
(2.12)
\]
The same data as listed in Table 1 are used to testify the solution based on barycentric coordinates. The numerical results are given in Table 3, which shows a complete coherence with Table 2.

**Table 3: Solution of “pseudo 4P” in barycentric coordinates**

<table>
<thead>
<tr>
<th>Two possible biases ($x_4$):</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4^+ = -100.0006$</td>
</tr>
<tr>
<td>$x_4^- = -57479918.164$</td>
</tr>
</tbody>
</table>

Two possible sets of barycentric coordinates for the user receiver position:

<table>
<thead>
<tr>
<th>$\lambda_1^+$</th>
<th>$\lambda_2^+$</th>
<th>$\lambda_3^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.671$</td>
<td>$-1.033$</td>
<td>$-2.008$</td>
</tr>
<tr>
<td>$5.760$</td>
<td>$-3.027$</td>
<td>$-4.387$</td>
</tr>
</tbody>
</table>

The admissible barycentric coordinates of the user receiver:

<table>
<thead>
<tr>
<th>$\lambda_1^+$</th>
<th>$\lambda_2^+$</th>
<th>$\lambda_3^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.671$</td>
<td>$-1.033$</td>
<td>$-2.008$</td>
</tr>
</tbody>
</table>

The admissible geocentric coordinates of the user receiver:

<table>
<thead>
<tr>
<th>$x_1^+$</th>
<th>$x_2^+$</th>
<th>$x_3^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1111590.460$</td>
<td>$-4348258.631$</td>
<td>$4527351.820$</td>
</tr>
</tbody>
</table>
Literature


Grafarend, E. W. and J. Shan (1996a): Closed-form solution of P4P or the three-dimensional resection problem in terms of Möbius barycentric coordinates, J. Geodesy, in print

Grafarend, E. W. and J. Shan (1996b): Closed-form solution to the twin P4P or the combined three-dimensional resection - intersection problem in terms of Möbius barycentric coordinates, J. Geodesy, to appear


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