ESA living planet symposium
28 June – 2 July 2010, Bergen, Norway

GOCE data analysis:
realization of the invariants approach
in a high performance computing environment

Oliver Baur, Nico Sneeuw, Jianqing Cai, Matthias Roth

Institute of Geodesy
University of Stuttgart
Outline

- GOCE gradiometry
- Invariants representation
- Linearization
- Synthesis of unobserved GGs
- Stochastic model
- High performance computing
- Summary
GOCE gradiometry

\[ \Gamma = -V + \Omega^2 + \dot{\Omega} \quad \rightarrow \text{separation of centrifugal and Euler effects} \]

(\text{star tracker, gradiometer})

\[ \Gamma \quad \text{... observation tensor} \]
\[ V \quad \text{... gravitational tensor} \]
\[ \Omega^2 \quad \text{... centrifugal tensor} \]
\[ \dot{\Omega} \quad \text{... Euler tensor} \]

gravitational gradients

\[ V = V_{ij} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12} & V_{22} & V_{23} \\ V_{13} & V_{23} & V_{33} \end{bmatrix} , \quad V_{11} + V_{22} + V_{33} = 0 \]
**GOCE gradiometry**

High-sensitive axes accuracy: $10^{-12}$ m s$^{-2}$

Low-sensitive axes accuracy: $10^{-9}$ m s$^{-2}$

High-accurate determination of $V_{11}, V_{22}, V_{33}, V_{13}$

ESA living planet symposium 2010
Invariants representation

**standard approach:**

observations: $V_{ij}$

gradiometer orientation essential (tensor transformation)

**alternative approach:**

observations: $J = J\{V\} = J\{V_{ij}\}$

no orientation information required

→ rotational invariants
Invariants representation

- rotational invariant \( J\{V^G\} = J\{V^M\} \)
- complete system (minimal basis) consists of three independent invariants

\[
\begin{align*}
\text{system I} & & \text{system II} & & \text{system III} \\
J_1 &= \text{tr } V & I_1 &= \text{tr } V & \Lambda_1 \\
J_2 &= \text{tr } V^2 & I_2 &= 0.5\left(\text{tr } V)^2 - \text{tr } V^2\right) & \Lambda_2 \\
J_3 &= \text{tr } V^3 & I_3 &= \det V & \Lambda_3
\end{align*}
\]

- Waring formula
- Newton-Girard formula
- characteristic equation

\[
p(\Lambda) = \det(\mathbf{V} - \Lambda \mathbf{I}_3) = 0
\]

\[
\Lambda^3 - I_1\Lambda^2 + I_2\Lambda - I_3 = 0
\]
invariants system of a symmetric, trace-free second-order tensor in $\mathbb{R}^3$

\[ I_1 = 0 \]
\[ I_2 = -\frac{1}{2}(V_{11}^2 + V_{22}^2 + V_{33}^2) - V_{12} - V_{13} - V_{23}^2 \]
\[ I_3 = V_{11}V_{22}V_{33} + 2V_{12}V_{13}V_{23} - V_{11}V_{23}^2 - V_{22}V_{13}^2 - V_{33}V_{12}^2 \]

- analysis of $I_1$ provides the trivial solution ($\rightarrow$ constraints)
- non-linear gravity field functionals
- gravitational gradients products
- mixing of gravitational gradients
Invariants representation

reference gravitational gradients in the GRF

reference invariants

ESA living planet symposium 2010
### Invariants representation

#### Pros and cons of the gravitational gradients tensor invariants approach

<table>
<thead>
<tr>
<th>Pros</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar-valued functionals</td>
<td>non-linear observables</td>
</tr>
<tr>
<td>independent of the gradiometer orientation in space</td>
<td>gravitational gradients required with compatible accuracy</td>
</tr>
<tr>
<td>independent of the orientation accuracy</td>
<td>(full tensor gradiometry)</td>
</tr>
<tr>
<td>independent of reference frame rotations / parameterization</td>
<td>huge computational costs, iterative parameter estimation</td>
</tr>
<tr>
<td></td>
<td>stochastic model of invariants</td>
</tr>
</tbody>
</table>
Invariants representation

Pros and cons of the gravitational gradients tensor invariants approach

Pros

- scalar-valued functionals
- independent of the gradiometer orientation in space
- independent of the orientation accuracy
- independent of reference frame rotations / parameterization

Cons

- linearization
- synthesis of inaccurate gravitational gradients
- high performance computing
- error propagation

ESA living planet symposium 2010
gravitational gradients: \[ V_{ij}(\lambda, \varphi, r) = \sum_{l=0}^{L} \sum_{m=0}^{l} f(\lambda, \varphi, r; c_{lm}, s_{lm}) \]

invariants:

\[ I_2 = \sum V_{i,j_1} V_{i,j_2} \]
\[ I_3 = \sum V_{i,j_1} V_{i,j_2} V_{i,j_3} \]

linearization (calculation of perturbations): \[ V_{ij} = U_{ij} + T_{ij} \]

additional effort (per iteration): synthesis of \[ U_{ij} \] up to \( L^{ref} \leq L \)
dependence on maximum resolution $L_{\text{ref}}$ of reference field

- GGM signal
- DE-RMS values relative to GGM:
- analysis $V_{33}$
Linearization

dependence on maximum resolution $L_{\text{ref}}$ of reference field

- GGM signal
- DE-RMS values relative to GGM:
  - analysis $V_{33}$
- DE-RMS values relative to $V_{33}$:
  - 1$\text{st}$ it. analysis $I_2, L_{\text{ref}} = 0$ (GRS80)
  - 2$\text{nd}$ it. analysis $I_2, L_{\text{ref}} = 0$
  - 3$\text{rd}$ it. analysis $I_2, L_{\text{ref}} = 0$

![Graph showing linearization and DE-RMS values](graph.png)

ESA living planet symposium 2010
Linearization

dependence on maximum resolution $L^{\text{ref}}$ of reference field

- GGM signal
- DE-RMS values relative to GGM:
  - analysis $V_{33}$
- DE-RMS values relative to $V_{33}$:
  - 1st it. analysis $I_2$, $L^{\text{ref}} = 0$ (GRS80)
  - 1st it. analysis $I_2$, $L^{\text{ref}} = 2$ (GRS80)
dependence on maximum resolution $L^\text{ref}$ of reference field

- GGM signal

DE-RMS values relative to GGM:
- analysis $V_{33}$

DE-RMS values relative to $V_{33}$:
- 1st it. analysis $I_2$, $L^\text{ref} = 0$ (GRS80)
- 1st it. analysis $I_2$, $L^\text{ref} = 2$ (GRS80)
- 1st it. analysis $I_2$, $L^\text{ref} = 200$ (OSU86F)

ESA living planet symposium 2010
Linearization

dependence on maximum resolution $L_{\text{ref}}$ of reference field

- GGM signal

DE-RMS values relative to GGM:
- analysis $V_{33}$

DE-RMS values relative to $V_{33}$:
- 1st it. analysis $I_2$, $L_{\text{ref}} = 0$ (GRS80)
- 1st it. analysis $I_2$, $L_{\text{ref}} = 2$ (GRS80)
- 1st it. analysis $I_2$, $L_{\text{ref}} = 200$ (OSU86F)
- 2nd it. analysis $I_2$

ESA living planet symposium 2010
Linearization

dependence on maximum resolution $L_{\text{ref}}$ of reference field

**conclusions**
- small linearization error
- fast convergence
- numerically efficient
- insensitive towards linearization field
Synthesis of unobserved GGs

- invariants representation requires the GGs with compatible accuracy (full tensor gradiometry)
- GOCE: $V_{12}$ and $V_{23}$ highly reduced in accuracy
- synthetic calculation of inaccurate GGs (forward modeling)
- avoid a priori information to leak into gravity field estimate
- $V_{12}, V_{23} \ll V_{11}, V_{22}, V_{33} \rightarrow$ minor influence

additional effort (per iteration): synthesis of $V_{12}$ and $V_{23}$
Synthesis of unobserved GGs

\[ I_2 = \frac{1}{2} \left( V_{11}^2 + V_{22}^2 + V_{33}^2 \right) \]

\[ -V_{12}^2 - V_{13}^2 - V_{23}^2 \]
Synthesis of unobserved GGs

\[ I_2 = -\frac{1}{2} \left( V_{11}^2 + V_{22}^2 + V_{33}^2 \right) - V_{12}^2 - V_{13}^2 - V_{23}^2 \]
Synthesis of unobserved GGs

impact of GGs synthesis on the estimation of geopotential parameters

- GGM signal
- DE-RMS values relative to GGM:
  - analysis $V_{33}$
- DE-RMS values relative to $V_{33}$:
  - 2\textsuperscript{nd} it. analysis $I_2$
  - 1\textsuperscript{st} it. analyse $I_2$, OSU86F

 ESA living planet symposium 2010
Synthesis of unobserved GGs

impact of GGs synthesis on the estimation of geopotential parameters

conclusions

- full tensor reconstructed
- no a priori information needed
- no impact on overall convergence
- numerically efficient
Stochastic model

linearized invariant

\[ \delta I_2 = c_1 T_{11} + c_2 T_{12} + c_3 T_{13} + c_4 T_{22} + c_5 T_{23} + c_6 T_{33} \]

linearized overall functional model

\[
\begin{bmatrix}
\delta I_{2}^1 \\
\vdots \\
\delta I_{2}^n
\end{bmatrix} =
\begin{bmatrix} c_1 & 0 \\
\vdots & \ddots \\
0 & c_1^n \\
\end{bmatrix}
\begin{bmatrix} T_{11}^1 \\
\vdots \\
T_{11}^n
\end{bmatrix} + \cdots +
\begin{bmatrix} c_6 & 0 \\
\vdots & \ddots \\
0 & c_6^n \\
\end{bmatrix}
\begin{bmatrix} T_{33}^1 \\
\vdots \\
T_{33}^n
\end{bmatrix}
\]

stochastic model of gravitational gradients

\[
D(T) =
\begin{bmatrix}
D(T_{11}) & C(T_{11}, T_{12}) & \cdots & C(T_{11}, T_{33}) \\
C(T_{12}, T_{11}) & D(T_{12}) & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\text{sym.} & \cdots & D(T_{33})
\end{bmatrix}
\]
correlations neglected

\[ c_1 = -U_{11}, \]
\[ c_2 = -2U_{12}, \]
\[ c_3 = -2U_{13}, \]
\[ c_4 = -U_{22}, \]
\[ c_5 = -2U_{23}, \]
\[ c_6 = -U_{33}. \]
Stochastic model

linearized invariant

$$\delta I_2 = c_1 T_{11} + c_2 T_{12} + c_3 T_{13} + c_4 T_{22} + c_5 T_{23} + c_6 T_{33}$$

linearized overall functional model

$$\begin{bmatrix}
\delta I_2^1 \\
\vdots \\
\delta I_2^n
\end{bmatrix} =
\begin{bmatrix}
c_1 & 0 \\
\vdots & \ddots \\
0 & c_1^n
\end{bmatrix}
\begin{bmatrix}
T_{11}^1 \\
\vdots \\
T_{11}^n
\end{bmatrix} + \cdots +
\begin{bmatrix}
c_6 & 0 \\
\vdots & \ddots \\
0 & c_6^n
\end{bmatrix}
\begin{bmatrix}
T_{33}^1 \\
\vdots \\
T_{33}^n
\end{bmatrix}$$

stochastic model of gravitational gradients

$$\mathbf{D}(\mathbf{T}) =
\begin{bmatrix}
\mathbf{D}(T_{11}) & 0 \\
\mathbf{D}(T_{12}) & \ddots \\
0 & \mathbf{D}(T_{33})
\end{bmatrix}$$

$$c_1 = -U_{11}$$
$$c_2 = -2U_{12}$$
$$c_3 = -2U_{13}$$
$$c_4 = -U_{22}$$
$$c_5 = -2U_{23}$$
$$c_6 = -U_{33}$$
Stochastic model

error propagation

\[ \mathbf{D}(T_{ij}) = (\mathbf{F}_{Vj}^T \mathbf{F}_{Vj})^{-1} \]

\[ \mathbf{D}(I_2) = \mathbf{J}_1 \mathbf{F}_{V_{11}}^{-1} (\mathbf{J}_1 \mathbf{F}_{V_{11}}^{-1})^T + \ldots + \mathbf{J}_6 \mathbf{F}_{V_{33}}^{-1} (\mathbf{J}_6 \mathbf{F}_{V_{33}}^{-1})^T \]

matrices of linear factors (Jacobian)

\[ \mathbf{J}_k = \text{diag}[c_k^1 \cdots c_k^n], \quad k = 1, \ldots, 6 \]

**Conclusion**

- invariants variance-covariance matrix by products between the (diagonal) matrices of linear factors and the inverse gravitational gradients filter matrices
High performance computing

- “brute-force” normal equations system “inversion”
- splitting the computational effort on several CPUs
- parallelization using OpenMP, MPI or OpenMP+MPI

- computation platforms provided by the High Performance Computing Centre Stuttgart (HLRS)
  - NEC SX-9 (array processor)
  - 12 nodes, 192 CPUs
  - TPP: 19.2 TFlops
High performance computing

- shared memory systems
  - parallelization via OpenMP
  - block-wise design matrix assembly
  - successive normal equations system assembly
  - algebraic operations by BLAS routines
  - normal equations system “inversion” by Cholesky decomposition
  - LAPACK routines
High performance computing

- parallelization via MPI
- block-wise design matrix assembly
- successive normal equations system assembly
  → block-cyclic data distribution
- algebraic operations by PBLAS routines
- normal equations system “inversion” by Cholesky decomposition
- ScaLAPACK routines
High performance computing

<table>
<thead>
<tr>
<th># CPUs</th>
<th>1</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>design matrix assembly (%)</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>NES assembly (%)</td>
<td>93.8</td>
<td>92.1</td>
<td>88.8</td>
</tr>
<tr>
<td>NES inversion (%)</td>
<td>1.5</td>
<td>3.4</td>
<td>6.8</td>
</tr>
<tr>
<td>speed-up (-)</td>
<td>1</td>
<td>3.8</td>
<td>7.2</td>
</tr>
</tbody>
</table>

- good runtime scaling on shared memory architectures limited to ~8 CPUs
- normal equations system assembly most time-consuming part

conclusion
- parallelization performed successfully

ESA living planet symposium 2010
the invariants approach is an independent alternative for SGG analysis to more conventional methods

independent of reference frame rotations and the gradiometer frame orientation in space

efficient linearization: small linearization error, fast convergence

full tensor gradiometry reconstruction by synthesis of unobserved gravitational gradients

approaches for the stochastic model handling of invariants

algorithms successfully implemented on high performance computing platforms

analysis of simulated GOCE data demonstrates the invariants approach to be a viable method for gravity field recovery

application on real data

Summary

ESA living planet symposium 2010