Reference
ellipsoidal gravity potential field
and
gravity intensity field
of
degree/order 360/360

(Manual of using
ellipsoidal harmonic coefficients
“ellipfree.dat” and “ellipmean.dat”)

Alireza A. Ardalan and Erik W. Grafarend

Department of Geodesy
and GeoInformatics
Stuttgart University
Geschwister-Scholl Str. 24D
70174 Stuttgart
Germany

e-mail: grafarend@gis.uni-stuttgart.de
1. Reference potential field of the external gravity field of the Earth

For a rigid uniformly rotating earth, the gravity potential field \( w(\lambda, \phi, \eta) \) can be additively decomposed into the gravitational potential field \( u(\lambda, \phi, \eta) \) and the centrifugal potential field \( v(\lambda, \phi, \eta) \), namely
\[
w(\lambda, \phi, \eta) = u(\lambda, \phi, \eta) + v(\lambda, \phi, \eta) .
\] (1.1)

See Appendix A, for the definition of ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \).

The multiplicative decomposition of the gravitational potential field into separable functions \( u(\lambda, \phi, \eta) = \Lambda(\lambda) \Phi(\phi) H(\eta) \) generates the following eigenvalue solution of the three dimensional Laplace partial differential equation.

\[
u(\lambda, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{nm}(\lambda, \phi, \eta) \frac{Q_{n+1}(\sin \eta)}{Q_n(\sin \eta)} e_n(\lambda, \phi)
\] (1.2)

where \( e_n(\lambda, \phi) \) are surface ellipsoidal harmonic functions
\[
e_n(\lambda, \phi) = P^m_n(\sin \phi) \left( \frac{\cos m \lambda}{\sin m \lambda} \right)
\] (1.3)

(1.2) is valid for the space \( \{\mathbb{R}^3 / E^2_{u,b}\} \) which is external to the reference ellipsoid of \((x^2 + y^2) / \varepsilon^2 \cos^2 \eta_b + z^2 / \varepsilon^2 \sin^2 \eta_b = 1\). The normalised associated Legendre functions of the first kind \( P^m_n(\sin \phi) \) and of the second kind \( Q^m_n(\sin \eta) \) are defined in Appendix B. The surface ellipsoidal harmonic functions \( e_n(\lambda, \phi) \) are orthonormal with respect to the weighted scalar product
\[
\langle e_p(\lambda, \phi) e_q(\lambda, \phi) \rangle = \delta_{pq} \delta_{nm},
\] (1.4)

where the weight function is defined by
\[
w(\phi) = \frac{a}{\sqrt{b^2 + \varepsilon^2 \sin^2 \phi}} \left( \frac{1}{2} + \frac{1}{4 \varepsilon} \ln \frac{a + \varepsilon}{a - \varepsilon} \right),
\] (1.5)

\( S \) is the area of the surface of the reference ellipsoid \( E^2_{u,b} \)
\[
S = area (E^2_{u,b}) = 4\pi a \left( \frac{1}{2} + \frac{1}{4 \varepsilon} \ln \frac{a + \varepsilon}{a - \varepsilon} \right).
\] (1.6)

The representation of the centrifugal potential in terms of (i) Cartesian coordinates \( \{x, y, z\} \), (ii) ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \) and (iii) surface ellipsoidal harmonic functions \( e_n(\lambda, \phi) \) are as follows.

(i) \( v(x, y) = \frac{1}{2} \omega^2(x^2 + y^2) \) \hspace{2cm} (1.7)

(ii) \( v(\phi, \eta) = \frac{1}{2} \omega^2 \varepsilon^2 \cos^2 \eta \cos^2 \phi \) \hspace{2cm} (1.8)
\[
\begin{align*}
\psi(\phi, \eta) &= \frac{1}{3} \omega^2 \left( \frac{P_{20}^*}{P_{20}}(\sinh \eta) \right) \varepsilon^2 \cos^2 \phi \\
&= \frac{2}{9} \omega^2 \left( \frac{P_{20}^*}{P_{20}}(\sinh \eta) \right) \varepsilon^2 e_{20}
\end{align*}
\]  

(10)

where to derive (10) we have used the following relations

\[
\cos^2 \phi = \frac{2}{3}(P_{20}^*(\sin \theta) - \frac{1}{\sqrt{y}} P_{20}^*(\sin \theta))
\]

(11)

If the series of ellipsoidal harmonic expansion of the earth be extended up to a limited degree and order, one arrives at an approximate representation for the external gravitational potential field of the earth, which can be used as a reference gravitational field. For example if we expand the ellipsoidal harmonic expansion of the external gravitational field of the earth up to degree order $360/360$ we have

\[
U(\lambda, \phi, \eta) = \sum_{n=0}^{360} \sum_{m=-n}^{n} u_{nm} \frac{Q_{nm}^*}{Q_{nm}^*} e_{nm} (\lambda, \phi)
\]  

(12)

with

\[
e_{nm} (\lambda, \phi) = P_{nm}^*(\sin \theta) \left[ \cos m \lambda \right] \sin m | \lambda | \forall m \geq 0
\]

(13)

which is an approximate solution of the actual eigen-value problem of the three dimensional Laplace partial differential equation. The ellipsoidal harmonic coefficients $u_{nm}$ appearing in (12) can be determined either, through an ellipsoidal harmonic analysis of the external gravitational field of the earth, or by an exact transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients. In the Section 3, we will see the numerical details of exact transformation of spherical harmonic coefficients into ellipsoidal ones.

2. Reference gravity intensity field of the external gravity field of the Earth

In the preceding section the following formulation was presented for external gravitational field of the earth

\[
u(\lambda, \phi, \eta) = \sum_{n=0}^{360} \sum_{m=-n}^{n} u_{nm} \frac{Q_{nm}}{Q_{nm}^*} e_{nm}(\lambda, \phi).
\]  

(2.1)

Gradient of formula (2.1) can provide us with a presentation for gravitational intensity field for the external space of the earth.

\[
\text{Grad} \nu(\lambda, \phi, \eta) = \text{Grad} \left( \sum_{n=0}^{360} \sum_{m=-n}^{n} u_{nm} \frac{Q_{nm}}{Q_{nm}^*} e_{nm}(\lambda, \phi) \right)
\]

(2.2)

where the metric tensor coefficients are as follows

\[
g_{11} = \varepsilon^2 \cos^2 \eta \cos^2 \phi
\]

(2.3)

\[
g_{22} = g_{33} = \varepsilon^2 (\cosh^2 \eta - \cos^2 \phi)
\]

(2.4)
Box 2-1: Partial derivatives of the gravitational potential of the external gravitational field of the earth.

\[
\frac{\partial u(\lambda, \phi, \eta)}{\partial \lambda} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{Q_{n}^{m}(\sinh \eta)}{Q_{n}^{m}(\sinh \eta_{b})} \frac{\partial e_{n,m}(\lambda, \phi)}{\partial \lambda}
\]

(2.5)

\[
\frac{\partial u(\lambda, \phi, \eta)}{\partial \phi} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{Q_{n}^{m}(\sinh \eta)}{Q_{n}^{m}(\sinh \eta_{b})} \frac{\partial e_{n,m}(\lambda, \phi)}{\partial \phi}
\]

(2.6)

\[
\frac{\partial u(\lambda, \phi, \eta)}{\partial \eta} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{dQ_{n}^{m}(\sinh \eta)}{d\eta} \frac{\partial e_{n,m}(\lambda, \phi)}{\partial \eta}
\]

(2.7)

subject to

\[
\frac{\partial e_{n,m}(\lambda, \phi)}{\partial \lambda} = P_{n}^{m}(\sin \phi) \left[ -m \sin m \lambda \right] \forall m \geq 0
\]

\[
\frac{\partial e_{n,m}(\lambda, \phi)}{\partial \phi} = \frac{dP_{n}^{m}(\sin \phi)}{d\phi} \left[ \cos m \lambda \right] \forall m \geq 0
\]

(2.8)

Box 2-2: First order derivatives of the Legendre functions of the first kind and the Legendre functions of second kind (N. Thong, 1989).

(i) Non-recursive formulas for computations of the first order derivatives of the Legendre functions of the first kind:

\[
\frac{dP_{0}^{m}(\sin \phi)}{d\phi} = 0,
\]

\[
\frac{dP_{n}^{m}(\sin \phi)}{d\phi} = -\sqrt{\frac{n(n+1)}{2}} P_{n-1}^{m}(\sin \phi) \quad \forall n > 1
\]

(2.10)

\[
\frac{dP_{1}^{m}(\sin \phi)}{d\phi} = P_{1}^{m}(\sin \phi) = \sqrt{3} \sin \phi
\]

(ii) Non-recursive formulas for computations of the first order derivatives of the Legendre functions of the second kind:

\[
\frac{dQ_{0,0}^{m}(\sinh \eta)}{d\eta} = -\frac{\cosh \eta_{b}}{\cosh \eta}
\]
\[
\frac{dQ^*_{m,0}(\sinh \eta)}{d\eta} = n \tanh \eta Q^*_{m,0}(\sinh \eta)
\]
\[
-(2n + 1) \frac{\cosh \eta}{\cosh \eta} Q^*_{m-1,0}(\sinh \eta)
\]
\[
\frac{dQ^*_{m,0}(\sinh \eta)}{d\eta} = -(n - m + 1) Q^*_{m,0}(\sinh \eta)
\]
\[-m \tanh \eta Q^*_{m,0}(\sinh \eta) \quad \forall m \in \{1, n\}
\]
\[
\frac{dQ^*_{m,0}(\sinh \eta)}{d\eta} = n \tanh \eta Q^*_{m,0}(\sinh \eta)
\]
\[
-(2n + 1) \frac{\cosh \eta}{\cosh \eta} Q^*_{m-1,0}(\sinh \eta)
\]
\[
\frac{dQ^*_{m,0}(\sinh \eta)}{d\eta} = -(n - m + 1) Q^*_{m,0}(\sinh \eta)
\]
\[-m \tanh \eta Q^*_{m,0}(\sinh \eta) \quad \forall m \in \{1, n\}
\]
\[
\frac{dP^*_{n,m}(\sin \phi)}{d\phi} = \sqrt{\frac{n+1}{2}} - n(n-1) P^*_{n,m-1}(\sin \phi) \quad \forall n > 2
\]
\[
\frac{dP^*_{n,m}(\sin \phi)}{d\phi} = \sqrt{\frac{n+1}{2}} - n(n-1) P^*_{n,m-1}(\sin \phi) \quad \forall n > 2
\]
\[
\frac{dP^*_{n,m}(\sin \phi)}{d\phi} = \sqrt{\frac{n+1}{2}} P^*_{n,m}(\sin \phi)
\]
\[
-\sqrt{\frac{n+1}{2}} P^*_{n,0}(\sin \phi) \quad \forall n > 2
\]
\[
\frac{dP^*_{n,m}(\sin \phi)}{d\phi} = \sqrt{\frac{n+1}{2}} - m(m-1) P^*_{n,m-1}(\sin \phi)
\]
\[
-\sqrt{\frac{n+1}{2}} - m(m+1) P^*_{n,m+1}(\sin \phi) \quad \forall n, m > 2 \text{ and } m \leq n - 1
\]

In the same way, the centrifugal intensity can be determined from the gradient of the centrifugal potential.

\[
\text{Grad}(v(\phi, \eta)) = \text{Grad} \left( \frac{1}{2} \omega^2 \varepsilon^2 \cosh \eta \cos^2 \phi \right)
\]
\[
= \frac{1}{\sqrt{g_{22}}} \frac{\partial v(\lambda, \phi, \eta)}{\partial \phi} \mathbf{e}_\phi + \frac{1}{\sqrt{g_{23}}} \frac{\partial v(\lambda, \phi, \eta)}{\partial \eta} \mathbf{e}_\eta,
\]

(2.12)

The partial derivatives of the centrifugal potential with respect to \( \phi \) and \( \lambda \) are given in Box 2-3 below.

**Box 2-3: Partial derivatives of the centrifugal potential.**

\[
\frac{\partial v(\lambda, \phi, \eta)}{\partial \phi} = -\omega \varepsilon^2 \cosh \eta \sin \phi \cos \phi
\]

(2.13)

\[
\frac{\partial v(\lambda, \phi, \eta)}{\partial \eta} = \omega \varepsilon^2 \cosh \eta \sinh \eta \cos^2 \phi
\]

(2.14)

Finally, the sum of the gravitational intensity and centrifugal intensity gets the gravity intensity vector \( \gamma(\lambda, \phi, \eta) \) as follows:

\[
\gamma(\lambda, \phi, \eta) = \text{Grad} \left( u(\lambda, \phi, \eta) \right) + \text{Grad}(v(\phi, \eta))
\]

(2.15)

where \( \text{Grad}(u(\lambda, \phi, \eta)) \) and \( \text{Grad}(v(\phi, \eta)) \) are given by (2.2) and (2.12), respectively.
If the summation $(2.2)$ is continued up to some finite degree/order one arrives at an approximate representation of the external gravitational intensity vector field of the earth.

3. **Transformation of Spherical Harmonic Coefficients into Ellipsoidal Harmonic Coefficients**

Nowadays it is a common practice to represent the “Standard Gravity Earth Models” in terms of spherical harmonics. Fortunately, precise transformation relations between spherical and ellipsoidal harmonic coefficients are available and therefore one can transfer the spherical harmonic coefficients into ellipsoidal ones without any loss of accuracy. Box 3-1 offers a summary of the transformation formulae of spherical harmonic coefficients into ellipsoidal harmonic coefficients according to C. Jekeli (1981, 1988). In conjunction with the ellipsoidal harmonics, contributions by D. Gleason (1988, 1989), G. Sona (1996) and J. Yu and H. Cao (1996) should also be acknowledged.

**Box 3-1: Transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients**

Spherical harmonic coefficients, $u_{n,m}(\text{sphere})$, can be uniquely transformed into ellipsoidal harmonic coefficients, $u_{n,m}(\text{ellipsoid})$ via

$$u_{n,m}(\text{ellipsoid}) = Q_{n,l}^* (\sin h \eta_h) \sum_{l=0}^{(n-m)/2} \lambda_{n,m,l} u_{n,m,l}(\text{sphere})$$  \hspace{1cm} (3.1)

$$\lambda_{n,m,l} = \frac{(2n-2l)!n!}{(2n)!}(\frac{n-m}{n!}) \left( \frac{2(n-4l+1)(n-m)(n+m)!}{(2n+1)(n-2l+m)(n-2l-m)!} \right)^{1/2} \left( \frac{\xi}{a} \right)^3$$ \hspace{1cm} (3.2)

$$\forall n \in [0,\infty) \text{ and } m \in [-n, n]$$  \hspace{1cm} (3.3)

By expanding the factorials in (3.2), one can reach to the following recursive formula, which is numerically stable especially for high degree/orders.

$$\lambda_{n,m,l} = \frac{(2n-4l+1)(n-2l-m+1)(n-2l-m+2)}{(2n+1)(n-2l+m)(n-2l+m+2)}^{1/2}$$ \hspace{1cm} (3.4)

$$/ [2k(2n-2l+1)(2n-4l+3)^{1/2}] \times \left( \frac{\xi}{a} \right)^3 \lambda_{n,m,l-1}$$

$$\forall l \in [1, (n-m)/2]; n \in [0,\infty) \text{ and } m \in [-n, n]$$

with the start value

$$\lambda_{n,m,0} = 1 \forall n, m$$  \hspace{1cm} (3.5)

$Q_{n,l}^* (\sin h \eta_h)$ are associated Legendre functions of the second kind, see equation (B.20) for their corresponding recursive relations.

As one can read from (3.1)-(3.5) each of the ellipsoidal harmonic coefficients $u_{n,m}(\text{ellipsoid})$ is equal to the spherical harmonic coefficients of the same degree and order $u_{n,m}(\text{sphere})$ plus a linear combination
of spherical harmonic coefficients of the lower degree but the same order. There are three parameters involved in (3.1), namely linear eccentricity $e = \sqrt{a^2 - b^2}$, and the size parameters $\eta_0$ and $a$. In fact, two of these parameters, say $\{\eta_0, e\}$, are enough to determine the shape and size of a reference ellipsoid $\hat{\Sigma}_{\text{r}, \eta_0, e}$. The question now arises as to which reference ellipsoid these parameters are related. The size parameter $a$ in (3.2) can be identified from the identity $a \equiv R$, where $R$ is the scale factor normally given with the spherical harmonic coefficients. In fact, $R$ is the reference sphere $\Sigma^{2}_{\text{r}, R}$ out of which the spherical harmonic expansion is uniformly convergent. That is, it defines the validity space of spherical harmonic expansion of external gravitational field of the earth. The sphere $\Sigma^{2}_{\text{r}, R}$ in ellipsoidal harmonic expansion of external gravitational field of the earth is replaced by the reference ellipsoid $\hat{\Sigma}_{\text{r}, \eta_0, e}$. Similar to $\Sigma^{2}_{\text{r}, R}$, $\hat{\Sigma}_{\text{r}, \eta_0, e}$ defines the validity space of the ellipsoidal harmonic expansion. While $e \cosh \eta_0 = a$ is determined via the identity $a \equiv R$, the selection of $e \sinh \eta_0 = b$ should be determined according to second zonal spherical harmonic $J_{20}$. Especially when one is interested in a Somigliana-Pizzetti type reference ellipsoid as the validity space of ellipsoidal harmonic expansion. As one knows the reference ellipsoid of Somigliana-Pizzetti type is uniquely determined via four fundamental geodetic parameters $J_{20}$, $\Omega$, $W_6$, and $GM$ (see Grafarend and Ardalan, 1999).

In our case, the ellipsoidal harmonic coefficients are supplied from the transformation of the spherical harmonic coefficients of EGM96 (F. Lemoine et al., 1996 and 1998). Table 3-1 is a collection of some spherical harmonic coefficients of EGM96. The harmonic coefficients of EGM96 are compatible with the following model.

$$ U(l, b, r) = \frac{g}{r} \left[1 + \sum_{n=2}^{\infty} \frac{R}{r} \sum_{m=0}^{n} u_{n,m}(l, b) \right] $$

(3.6)

$\varepsilon_{n,m}(l, b)$ are surface spherical harmonics

$$ e_{n,m}(l, b) = P^{+}_{n+m}(\sin b) \left[ \frac{\cos ml}{\sin \frac{nl}{m}} \right] \forall m \geq 0 $\forall m < 0

(3.7)

In (3.6) $\{l, b, r\}$ are spherical coordinates of the computational point, $R = 6378136.3$ m is the scale factor which defines the radius of the reference sphere $\Sigma^{2}_{\text{r}, R}$ out of which the series expansion of (3.6) is uniformly convergent. The product of Newton gravitational constant and the mass of the earth in the EGM96 (F. Lemoine et al., 1998) model is $gm = 3986004415.8 m^2 s^{-2}$. The spherical harmonic coefficients of EGM96 are in tide free system. However, they can be transferred into mean-tide or zero-frequency permanent tide systems (see F. Lemoine et al., 1998).

**Table 3-1**: Some spherical harmonic coefficients of the EGM96 global geopotential model.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$u_{0,m}$</th>
<th>$u_{n,m}$</th>
<th>$\sigma_{u_{n,m}}$</th>
<th>$\sigma_{\sigma_{n,m}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0.484165371E-03</td>
<td>0.00000000E+00</td>
<td>0.35610E-10</td>
<td>0.00000E+00</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-0.18698763E-09</td>
<td>0.11952812E-08</td>
<td>0.10000E-29</td>
<td>0.10000E-29</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.243914352E-05</td>
<td>-0.14001668E-05</td>
<td>0.58379E-10</td>
<td>0.54353E-10</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.957254173E-06</td>
<td>0.00000000E+00</td>
<td>0.18094E-10</td>
<td>0.00000E+00</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.202998882E-05</td>
<td>0.248513158E-06</td>
<td>0.13965E-09</td>
<td>0.13645E-09</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.904627765E-06</td>
<td>-0.61902594E-06</td>
<td>0.10962E-09</td>
<td>0.11182E-09</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.721072657E-06</td>
<td>0.141345626E-05</td>
<td>0.95156E-10</td>
<td>0.93285E-10</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.539873863E-06</td>
<td>0.00000000E+00</td>
<td>0.10423E-09</td>
<td>0.00000E+00</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-0.55632161E-06</td>
<td>-0.47344026E-06</td>
<td>0.85674E-10</td>
<td>0.82408E-10</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.350691056E-06</td>
<td>0.662671572E-06</td>
<td>0.16000E-09</td>
<td>0.16390E-09</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.990771803E-06</td>
<td>-0.20092835E-06</td>
<td>0.84657E-10</td>
<td>0.82662E-10</td>
</tr>
</tbody>
</table>
Some of the computed ellipsoidal harmonic coefficients are represented in Table 3-2. These coefficients are compatible with the following series expansion, which is convergent outside the reference ellipsoid \( \tilde{H}_0 = a = 6378136.3 \text{ m} \) is coming from identity \( a = S^2_{m} R \), and \( \varepsilon \sinh \eta_0 = b = 6356751.647 \text{ m} \) is the linear eccentricity \( \varepsilon = 521.853.580 \text{ (m) of } WGD \; 2000 \text{ in tide free system (E. Gräfenarend and A. Ardalan, 1999)} \) which has been computed based on four fundamental eccentric parameters including the \( J_m \) coefficient of EGM96.

\[
U(\lambda, \phi, \eta) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} u_{n,m}(\sinh \eta) b_{n,m}(\sinh \eta) c_{n,m}(\lambda, \phi)
\]

where \( gm = 3.986004415E+8 \; \text{ m}^3 \text{ s}^{-2} \). \( b_{n,m}(\sinh \eta) \) are the numerically stabilised associated Legendre functions of the second kind (see (B.20)).

Table 3-2: Ellipsoidal harmonic coefficients; valid for the outer space of the reference ellipsoid \( \tilde{H}_0 = a = 6378136.3 \text{ m} \) is coming from identity \( a = S^2_{m} R \), and \( \varepsilon = 521.853.580 \text{ (m) of } WGD \; 2000 \text{ in tide free system (E. Gräfenarend and A. Ardalan, 1999)} \)

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>( u_{n,m} )</th>
<th>( u_{n,-m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.00111910296E+00</td>
<td>0.00000000000E+00</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000000000E+00</td>
<td>0.00000000000E+00</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>5.15993819297E-04</td>
<td>0.00000000000E+00</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1.87706137878E-010</td>
<td>1.19987290652E-009</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.44490809191E-006</td>
<td>-1.4032755310E-006</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>9.62981735654E-007</td>
<td>0.00000000000E+00</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2.04137403489E-006</td>
<td>2.49006947306E-007</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th></th>
<th></th>
<th>9.0868480195E-007</th>
<th>-6.21802123549E-007</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>7.2295762132E-007</td>
<td>1.41805355446E-006</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-2.52341695041E-007</td>
<td>0.0000000000E+000</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-5.4026343011E-007</td>
<td>-4.7691456602E-007</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3.59725179139E-007</td>
<td>6.6304016804E-007</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>9.9562081704E-007</td>
<td>-2.01911680303E-007</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>-1.89079400983E-007</td>
<td>3.09702607341E-007</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>7.49051604596E-008</td>
<td>0.0000000000E+000</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-5.1134335153E-008</td>
<td>-9.3867928503E-008</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>6.62145933033E-007</td>
<td>-3.28971299283E-007</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-4.52836882666E-007</td>
<td>-2.11878169088E-007</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>-2.96830197182E-007</td>
<td>4.99230202378E-008</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1.7546971171E-007</td>
<td>-6.7128841225E-007</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>9.7043570236E-009</td>
<td>-2.37883139282E-007</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>1.0950071286E-009</td>
<td>2.45140824272E-008</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>-1.24465828265E-007</td>
<td>1.20895792530E-007</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>-4.7892803820E-008</td>
<td>9.69535023612E-008</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1.00847319743E-007</td>
<td>-2.40885781042E-008</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>4.02735731045E-009</td>
<td>-1.20836917488E-008</td>
</tr>
<tr>
<td>36</td>
<td>36</td>
<td>4.61652960655E-009</td>
<td>-5.96190863784E-009</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>4.24468051530E-009</td>
<td>3.94284208953E-010</td>
</tr>
<tr>
<td>120</td>
<td>120</td>
<td>-4.58322787954E-010</td>
<td>-1.5665034569E-009</td>
</tr>
<tr>
<td>180</td>
<td>180</td>
<td>-4.07932875977E-010</td>
<td>-5.89692325818E-010</td>
</tr>
<tr>
<td>240</td>
<td>240</td>
<td>-3.3155374510E-010</td>
<td>-4.62401899146E-011</td>
</tr>
<tr>
<td>300</td>
<td>300</td>
<td>-5.04021159193E-011</td>
<td>-1.01615094489E-010</td>
</tr>
<tr>
<td>360</td>
<td>360</td>
<td>-4.49017687390E-025</td>
<td>-8.33010128012E-011</td>
</tr>
</tbody>
</table>

4. Instruction for using ellipsoidal harmonic coefficients

The file “ellifree.dat” contains the ellipsoidal harmonic coefficients, which are derived as explained in previous section in *tide-free* system. These coefficients are compatible with following model out side the reference ellipsoid $E_{ab}$ of WGD2000 in *tide-free* system, $a = \varepsilon \cosh \eta_b$, $b = \varepsilon \sinh \eta_b$, where $\eta_b = 3.194713538106130$ and $\varepsilon = 52.1853580$ (m) (see E. Graftendale and A. Ardalan, 1999).

$$U(\lambda, \phi, \eta) = \frac{gm}{6 \times 378} \times 136.3 \times \sum_{n=0}^{20} \sum_{m=-n}^{n} u_{nm} \frac{Q_{n,m}(\sinh \eta)}{Q_{n,m}(\sinh \eta_b)} \epsilon_{nm}(\lambda, \phi)$$

(3.9)

where $gm=3.986004.415E+8 m^3/s^2$. $Q_{n,m}(\sinh \eta)$ are the numerically stabilised *associated Legendre functions of the second kind* (see (B.20)).

(3.9) can be written in terms of Jacobi ellipsoidal coordinates $\{\lambda, \phi, \eta\}$ as follows

$$U(\lambda, \phi, \eta) = \frac{gm}{6 \times 378} \times 136.3 \times \sum_{n=0}^{20} \sum_{m=-n}^{n} u_{nm} \frac{Q_{n,m}^{(1)}(\xi)}{Q_{n,m}^{(1)}(\xi_b)} \epsilon_{nm}(\lambda, \phi)$$

(3.10)
the Jacobi ellipsoidal coordinates \( \{\lambda, \phi, u\} \) are defined in Appendix C and the associated Legendre functions of the second kind \( Q_m^-(u) \) are introduced in Appendix D.

The gravitational potential computed via the ellipsoidal harmonic coefficients of the file “ellipfree.dat” is in tide free system. However, if one is interested in computation of the gravitational potential field in mean-tide permanent-tide system then the coefficients presented in the file “ellipmean.dat” must be used. These coefficients are compatible with following model out side the reference ellipsoid \( E^2_{a,b} \) of WGD2000 in mean-tide permanent-tide system, \( a = \varepsilon \cosh \eta_h \), \( b = \varepsilon \sinh \eta_h \), where \( \eta_h = 3.194704493 \, 50831 \) and \( \varepsilon = 521858.317 \) (m) \( (\text{see E. Grefswend and A. Ardalan, 1999}) \)

\[
U(\lambda, \phi, \eta) = \frac{gm}{6 \, 378 \, 136.3} \sum_{n=0}^{+\infty} \sum_{m=-n}^{+n} a_{nm} \frac{Q_{m+1}^-(\sinh \eta)}{Q_{n+1}^-(\sinh \eta_h)} \tilde{c}_{nm}(\lambda, \phi) \tag{3.11}
\]

where \( gm=3 \times 10^4 004.415E+8 \, m^3 s^2 \). \( Q_{n+1}^-(\sinh \eta) \) are the numerically stabilised associated Legendre functions of the second kind \( (\text{see (D.1)}) \).

(3.11) can be written in terms of Jacobi ellipsoidal coordinates \( \{\lambda, \phi, u\} \) as shown by (3.10).
References:


Appendices A: Ellipsoidal Coordinates \( \{\lambda, \phi, \eta\} \)

In terms of ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \), a point in space can be located as the intersection of three coordinate surfaces. The coordinate surfaces are corresponding to three families of surfaces of the type (i) confocal oblate spheroids, (ii) confocal half hyperboloids, and (iii) planes. These families of surfaces are defined as follows.

(i) The family of confocal oblate spheroids
\[
\mathbb{E}^2_{\cosh \eta, \sinh \eta} := \left\{ x \in \mathbb{R}^3 \mid \frac{x^2 + y^2}{e^2 \cosh \eta} + \frac{z^2}{e^2 \sinh \eta} = 1, \ \eta \in (0, + \infty), \ e^2 := a^2 - b^2 \right\}
\]  
(A.1)

(ii) The family of confocal half hyperboloids
\[
\mathbb{H}^2_{\cosh \phi, \sinh \phi} := \left\{ x \in \mathbb{R}^3 \mid \frac{x^2 + y^2}{e^2 \cos^2 \phi} - \frac{z^2}{e^2 \sin^2 \phi} = 1, \ \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \ \phi \neq 0 \right\}
\]  
(A.2)

(iii) The family of half planes
\[
\mathbb{P}^2_{\tan \lambda, \lambda} := \left\{ x \in \mathbb{R}^3 \mid y = x \tan \lambda, \ \lambda \in [0, \pi] \right\}
\]  
(A.3)

Longitude \( \lambda \) gives orientation to the half planes. The latitude \( \phi \) is related to the inclination of the asymptotes of confocal half hyperboloids; the elliptic coordinate \( \eta \) in the form of \( e \cosh \eta \) defines the semi-minor axis of confocal oblate spheroids (confocal, oblate ellipsoids of revolution).

The forward and backward transformations of ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \) into Cartesian coordinates are collected in Box A.1. The Jacobi matrix of the forward transformation is reviewed in Definition A.1.

---

**Box A.1: Conversion of Cartesian coordinates \( \{x, y, z\} \) into ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \)**

**Forward transformation of ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \) into Cartesian coordinates \( \{x, y, z\} \)**

\[
x = e \cosh \eta \cos \phi \cos \lambda \\
y = e \cosh \eta \cos \phi \sin \lambda \\
z = e \sinh \eta \sin \phi
\]  
(A.4)

**Backward transformation of Cartesian coordinates \( \{x, y, z\} \) into ellipsoidal coordinates \( \{\lambda, \phi, \eta\} \)**

\[
\lambda = \begin{cases} 
\arctan \frac{y}{x} & \text{for } x > 0 \text{ and } y \geq 0 \\
\arctan \frac{y}{x} + \pi & \text{for } x < 0 \text{ and } y \neq 0 \\
\arctan \frac{y}{x} + 2\pi & \text{for } x > 0 \text{ and } y < 0 \\
\frac{\pi}{2} & \text{for } x = 0 \text{ and } y > 0 \\
3\frac{\pi}{2} & \text{for } x = 0 \text{ and } y < 0
\end{cases}
\]  
(A.5)

\[
\phi = \text{sgn}(z) \arcsin \left\{ \frac{1}{2e^2} \left[ e^2 - (x^2 + y^2 + z^2) \right] + \sqrt{(x^2 + y^2 + z^2 - e^2)^2 + 4e^4 e^2} \right\}^{1/2}
\]  
(A.6)
\[\eta = \text{arccosh}\left\{\frac{1}{2e^2}\left[x^2 + y^2 + z^2 + \varepsilon^2 + \sqrt{(x^2 + y^2 + z^2 + \varepsilon^2)^2 - 4\varepsilon^2(x^2 + y^2)}\right]\right\}^{1/2}\]  

(A.7)

Valid for

\[\lambda \in \{\lambda \in \mathbb{R} \mid 0 < \lambda < 2\pi\}\]

\[\phi \in \{\phi \in \mathbb{R} \mid -\frac{\pi}{2} < \phi < +\frac{\pi}{2}\}\]

\[\eta \in \{\eta \in \mathbb{R} \mid \eta > 0\}\]  

(A.8)

**Definition A-1:** Basic geometry of ellipsoidal coordinates \(\{\lambda, \phi, \eta\}\)

(i) *Jacobi matrix of transformation from the ellipsoidal coordinates \(\{\lambda, \phi, \eta\}\) into Cartesian coordinates \(\{x, y, z\}\)*

From equation (A.4) Jacobi matrix "\(J\)" of the transformation from ellipsoidal coordinates \(\{\lambda, \phi, \eta\}\) into Cartesian coordinates \(\{x, y, z\}\) can be constructed as follows

\[J := \begin{vmatrix} X_\lambda & X_\phi & X_\eta \\ Y_\lambda & Y_\phi & Y_\eta \\ Z_\lambda & Z_\phi & Z_\eta \end{vmatrix}\]  

(A.9)

(ii) *The partial derivatives involved in (A.9) read as*

\[X_\lambda = D_\lambda X = -\varepsilon \cosh \eta \cos \phi \sin \lambda\]

\[Y_\lambda = D_\lambda Y = \varepsilon \cosh \eta \cos \phi \cos \lambda\]

\[Z_\lambda = D_\lambda Z = 0\]

\[X_\phi = D_\phi X = -\varepsilon \cosh \eta \sin \phi \cos \lambda\]

\[Y_\phi = D_\phi Y = -\varepsilon \cosh \eta \sin \phi \sin \lambda\]

\[Z_\phi = D_\phi Z = \varepsilon \sinh \eta \cos \phi\]

\[X_\eta = D_\eta X = \varepsilon \sinh \eta \cos \phi \cos \lambda\]

\[Y_\eta = D_\eta Y = \varepsilon \sinh \eta \cos \phi \sin \lambda\]

\[Z_\eta = D_\eta Z = \varepsilon \cosh \eta \sin \phi\]

(iii) *The metric tensor*

\[G := J^T J\]

\[= \begin{vmatrix} \varepsilon^2 \cosh^2 \eta \cos^2 \phi & 0 & 0 \\ 0 & \varepsilon^2 (\cosh^2 \eta - \cos^2 \phi) & 0 \\ 0 & 0 & \varepsilon^2 (\cosh^2 \eta - \cos^2 \phi) \end{vmatrix}\]

\[= \begin{vmatrix} (g_{\lambda\lambda})^2 & 0 & 0 \\ 0 & (g_{\phi\phi})^2 & 0 \\ 0 & 0 & (g_{\eta\eta})^2 \end{vmatrix}\]  

(A.10)

(iv) *The Laplacian*

\[\Delta = \frac{1}{\sqrt{\varepsilon^2}} \left\{ \frac{\partial}{\partial \lambda} \left( \sqrt{\varepsilon^2} \frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial \phi} \left( \sqrt{\varepsilon^2} \frac{\partial}{\partial \phi} \right) + \frac{\partial}{\partial \eta} \left( \sqrt{\varepsilon^2} \frac{\partial}{\partial \eta} \right) \right\}\]

\[= \frac{1}{\varepsilon^2 (\sin^2 \phi + \sinh^2 \eta)} \left\{ \sin^2 \phi + \sinh^2 \eta \frac{\partial^2}{\partial \phi^2} - \tan \phi \frac{\partial}{\partial \phi} \right\}\]

\[+ \frac{\partial^2}{\partial \phi^2} + \tanh \eta \frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2}\]  

(A.11)
Appendix B: Normalised Associated Legendre Functions of the First and Second Kind

We define the normalised associated Legendre functions of the first kind \( P^*_m(\sin \phi) \) by the recurrence relations

\[
P^*_m(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2n}} \cos \phi P^*_{m,1,n-1}(\sin \phi)
\]

\[
P^*_n(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2(n-1)}} \cos \phi P^*_{n+1,n-1}(\sin \phi)
\]

\[
P^*_m(\sin \phi) = \frac{\sqrt{4n^2 - 1}}{\sqrt{2n} - m} \sin \phi P^*_{m-1,n,m}(\sin \phi) - \frac{(2n+1)(n + m - 1)(n - m - 1)}{\sqrt{2n - 1}^2} P^*_{n-2,m}(\sin \phi)
\]

subject to

\( \forall n \in [3, \infty) \) and \( m \in [0, n - 2] \)

with start up values

\[
P^*_0(\sin \phi) = 1
\]

\[
P^*_0(\sin \phi) = \sqrt{3} \sin \phi
\]

\[
P^*_1(\sin \phi) = \sqrt{3} \cos \phi
\]

\[
P^*_0(\sin \phi) = \frac{\sqrt{3}}{2} (3 \sin^2 \phi - 1)
\]

\[
P^*_1(\sin \phi) = \sqrt{3} \sin \phi \cos \phi
\]

\[
P^*_2(\sin \phi) = \frac{\sqrt{15}}{2} \cos^2 \phi
\]

Let us define the normalised associated Legendre functions of the first kind \( P^*_m(\sinh \eta) \) through an integral equation

\[
P^*_m(\sinh \eta) := i^m \int_0^\infty (\sinh \eta \cosh \eta \cos \phi)^m \cos m\phi d\phi
\]

where \( i = \sqrt{-1} \) is the imaginary unit. First few normalised associated Legendre functions of the first kind \( P^*_m(\sinh \eta) \) for \( n = 0, 1, 2 \) and \( m = 0 \) are as follows.

\[
P^*_0(\sinh \eta) = 1
\]

\[
P^*_1(\sinh \eta) = \sinh \eta
\]

\[
P^*_2(\sinh \eta) = \frac{1}{2} (3 \sinh^2 \eta + 1)
\]

The associated Legendre functions of the second kind can be defined by an integral relation of the type

\[
Q^*_m(\sinh \eta) = \frac{2^m}{(n + m)!} \int_0^\infty \cosh \eta \cos \phi \cosh \eta \cosh \tau \left( \frac{\sinh \eta}{\cosh \eta \cosh \tau} \right)^{n-\frac{m}{2}} d\tau
\]

with starting values for \( n = 0, 1, 2 \) and \( m = 0 \)

\[
Q^*_0(\sinh \eta) = \arccot(\sinh \eta)
\]
\[ Q^*_{s}(\sinh \eta) = 1 - \sinh \eta \arccot(\sinh \eta) \] (B.18)
\[ Q^*_{f}(\sinh \eta) = \frac{1}{2} \left(3\sinh^2 \eta + 1\right) \arccot(\sinh \eta) - 3 \sinh \eta \] (B.19)

Instead of the above integral formulas, in practice the associated Legendre functions of the second kind are better to be calculated via the recursive relations which enjoy the numerical stable, especially for the higher degree and order functions (N. Thong and E. Grafarend, 1989, G. Sona, 1996)

\[ Q^*_{\pm k}(\sinh \eta) = \sum_{n=\pm}^{\infty} Q^*_{\pm k}(\eta) \] (B.20)
\[ Q^*_{\pm k}(\eta) = \frac{(1 - n - m)(n + m)(n + 1)}{2(2n + 2k + 1)} \sinh^2 \eta \] (B.21)

\[ Q^*_{\pm k}(\eta) = \cosh^{n}\eta \left(\frac{\cosh \eta}{\sinh \eta}\right)^{n+1} \quad \forall \ n \in \mathbb{N}, \ m \in [-n,n] \subseteq \mathbb{Z} \] (B.22)

The summation (B.20) is continued until
\[ Q^*_{\pm \eta_{\text{max}}}(\eta) - Q^*_{\pm \eta_{\text{min}}}(\eta) < \sigma \] (B.23)

where \( \sigma \) indicates the numerical accuracy limit. For double precision accuracy, \( \sigma = 1 \times 10^{-16} \) may be adopted.

### Appendices C: Ellipsoidal Coordinates \( \{\lambda, \phi, u\} \)

In terms of ellipsoidal coordinates \( \{\lambda, \phi, u\} \), a point in space can be located as the intersection of the following family of surfaces.

(i) the family of confocal, oblate spheroids
\[
\mathbb{E}^2_{\varepsilon, \varepsilon^2} := \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ a \in (0, + \infty), \ e^2 := a^2 - b^2 \right\}
\] (C.1)

(ii) the family of confocal half hyperboloids
\[
\mathbb{H}^2_{\varepsilon, \varepsilon^2, \phi} := \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \ \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \ \phi = 0 \right\}
\] (C.2)

(iii) the family of half planes
\[
\mathbb{P}^2_{\varepsilon, \varepsilon^2, \phi} := \{ \mathbf{x} \in \mathbb{R}^3 \mid y = x \tan \lambda, \ \lambda \in [0, 2\pi] \}
\] (C.3)

The longitude \( \lambda \) gives orientation to the half planes. The latitude \( \phi \) is related to the inclination of the asymptotes of confocal half hyperboloids; the elliptic coordinate \( u \) coincides with the semi-minor axis of confocal oblate spheroids (confocal, oblate ellipsoids of revolution).

#### Box C-1: Conversion of Cartesian coordinates \( \{x, y, z\} \) into ellipsoidal coordinates \( \{\lambda, \phi, u\} \)

(i) Forward transformation from ellipsoidal coordinates \( \{\lambda, \phi, u\} \) into Cartesian coordinates \( \{x, y, z\} \)
\[
x = \sqrt{a^2 + \varepsilon^2 \cos^2 \phi \cos \lambda}
\]
\[
y = \sqrt{a^2 + \varepsilon^2 \cos \phi \sin \lambda}
\]
\[
z = u \sin \phi
\] (C.4)
(ii) Backward transformation of Cartesian coordinates \( \{x, y, z\} \) into ellipsoidal coordinates
\[
\lambda = \begin{cases} 
\arctan \frac{y}{x} & \text{for } x > 0 \text{ and } y \geq 0 \\
\arctan \frac{y}{x} + \pi & \text{for } x < 0 \text{ and } y \neq 0 \\
\arctan \frac{y}{x} + 2\pi & \text{for } x > 0 \text{ and } y < 0 \\
\frac{\pi}{2} & \text{for } x = 0 \text{ and } y > 0 \\
3\frac{\pi}{2} & \text{for } x = 0 \text{ and } y < 0 
\end{cases}
\]
\[\phi = (\text{sgn } z) \arcsin \left\{ \frac{1}{2\varepsilon^2} \left[ (x^2 + y^2 + z^2) \right]^{1/2} 
+ \left( (x^2 + y^2 + z^2 - \varepsilon^2) \right)^{1/2} + 4\varepsilon^2 x^2 \right\}^{1/2} \tag{C.6}\]
\[u = \frac{1}{2} \left[ (x^2 + y^2 + z^2 - \varepsilon^2) + (x^2 + y^2 + z^2) \right]^{1/2} \tag{C.7}\]

**Definition C-1:** Basic geometry of ellipsoidal coordinates \( \{\lambda, \phi, u\} \)

(i) **Jacobi matrix of the transformation from ellipsoidal coordinates \( \{\lambda, \phi, u\} \) into Cartesian coordinates \( \{x, y, z\} \)**

From equation (C.4) Jacobi matrix "J" of the transformation from ellipsoidal coordinates \( \{\lambda, \phi, u\} \) into Cartesian coordinates \( \{x, y, z\} \) can be constructed
\[
J := \begin{bmatrix}
X_\lambda & X_\phi & X_u \\
Y_\lambda & Y_\phi & Y_u \\
Z_\lambda & Z_\phi & Z_u
\end{bmatrix}
\] \tag{C.8}

The partial derivatives involved in (C.8) are as follows
\[
X_\lambda = D_\lambda X = -\sqrt{u^2 + \varepsilon^2} \cos \phi \sin \lambda \\
Y_\lambda = D_\lambda Y = \sqrt{u^2 + \varepsilon^2} \cos \phi \cos \lambda \\
Z_\lambda = D_\lambda Z = 0 \\
X_\phi = D_\phi X = -\sqrt{u^2 + \varepsilon^2} \sin \phi \cos \lambda \\
Y_\phi = D_\phi Y = -\sqrt{u^2 + \varepsilon^2} \sin \phi \sin \lambda \\
Z_\phi = D_\phi Z = u \cos \phi \\
X_u = D_u X = \frac{u}{\sqrt{u^2 + \varepsilon^2}} \cos \phi \cos \lambda \\
Y_u = D_u Y = \frac{u}{\sqrt{u^2 + \varepsilon^2}} \cos \phi \sin \lambda \\
Z_u = D_u Z = \sin \phi.
\]

(ii) **The metric tensor**
\[
dS^2 = [d\lambda, d\phi, du] J^T J \begin{bmatrix}
d\lambda \\
d\phi \\
du
\end{bmatrix}
\]
\[
G := J^T J = \begin{bmatrix}
(u^2 + \varepsilon^2) \cos^2 \phi & 0 & 0 \\
0 & u^2 + \varepsilon^2 \sin^2 \phi & 0 \\
0 & 0 & \frac{(u^2 + \varepsilon^2 \sin^2 \phi)/(u^2 + \varepsilon^2)}
\end{bmatrix}
\tag{C.9}
\]

\[
G := J^T J = \begin{bmatrix}
(u^2 + \varepsilon^2) \cos^2 \phi & 0 & 0 \\
0 & u^2 + \varepsilon^2 \sin^2 \phi & 0 \\
0 & 0 & \frac{(u^2 + \varepsilon^2 \sin^2 \phi)/(u^2 + \varepsilon^2)}
\end{bmatrix}
\tag{C.9}
\]

\[
G := J^T J = \begin{bmatrix}
(u^2 + \varepsilon^2) \cos^2 \phi & 0 & 0 \\
0 & u^2 + \varepsilon^2 \sin^2 \phi & 0 \\
0 & 0 & \frac{(u^2 + \varepsilon^2 \sin^2 \phi)/(u^2 + \varepsilon^2)}
\end{bmatrix}
\tag{C.10}
\]

16
Appendix D: Normalised Associated Legendre Functions of the Second Kind

The associated Legendre functions of the second kind can be calculated via following recursive relations which enjoy the numerical stability, especially for the higher degree and order functions

\[ Q_{\lambda\mu}^n(u) = \sum_{k=0}^{k_{\text{max}}} Q_{\lambda\mu}^k(u) \quad (\text{D.1}) \]

\[ Q_{\lambda\mu}^n(u) = \frac{\varepsilon^2(1-n|m|2k(n+|m|)+2k)}{2k(2n+2k+1)u^2} Q_{\lambda\mu}^{n-1}(u) \quad \forall \; k \geq 1 \quad (\text{D.2}) \]

\[ Q_{\lambda\mu}^n(u) = (u^2 + \varepsilon^2)^{1/2} \frac{(\frac{d}{du})^{n+1}}{u} Q_{\lambda\mu}^{n+1}(u) \quad \forall \; n \in N, \; m \in [-n,n] \subset Z \quad (\text{D.3}) \]

The summation (D.1) is continued until

\[ Q_{\lambda\mu}^n(u) - Q_{\lambda\mu}^{n-1}(u) < \sigma \quad (\text{D.4}) \]

where \( \sigma \) indicates the numerical accuracy limit. For double precision accuracy, \( \sigma = 1E-16 \) may be adopted.