Construction of directional wavelets on the sphere

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Geodätische Woche 2014
Motivation

- model refinement by localizing base functions
- tend to zero outside the area of influence
- model mainly data within the area of interest

Fig: Mascons (Lemoine, 2007)

Fig: Boundary elements (Weigelt, 2012)
Motivation

- model refinement by localizing base functions
- tend to zero outside the area of influence
- model mainly data within the area of interest
- very often: radial symmetric

![Radial base functions on a sphere](Fig: Radial base functions on a sphere)

![Mascons (Lemoine, 2007)](Fig: Mascons (Lemoine, 2007))

![Boundary elements (Weigelt, 2012)](Fig: Boundary elements (Weigelt, 2012))
Motivation

- observations by satellites have
  - preferred direction
  - converging of tracks

Fig: Spherical caps with radius 1.5° and points per cap
Idea

- isotropic functions $\psi(x, y)$ in spatial domain
  ($x$: location, $y$: node/center)
- linear transformation $\tilde{x} = E \cdot x$ and $\tilde{y} = E \cdot y$
- ‘elliptical’ contour lines per wavelets
Poisson wavelets

Poisson wavelets of order $N$:

$$\chi_n = \left( \frac{\partial}{\partial \|y\|} \frac{\|y\|}{\|x - y\|} \right)^n \frac{1}{\|x - y\|}$$

for $n = 0, 1, \ldots, N + 1$ and

$$\psi(x, y) = \frac{1}{4\pi R^2} \left( 2\chi_{N+1} + \chi_N \right)$$

recursive formulas up to $N = 9$

Fig: (Normalized) wavelet on the sphere

Fig: Cut along the meridian
Transformation

- keep size in North-South direction
- scaling in East-West direction
- empirical factor: 
  \[ f(\phi) := \exp \left( \frac{1}{2} - \frac{1}{2} \left( \frac{\phi}{45} \right)^2 \right) \]

Fig: Points within modified spherical caps and points per cap
‘Elliptical’ wavelets

\[
E^{-1} = (R_g^e)\top \begin{pmatrix}
1 & 0 & 0 \\
0 & f(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix} R_g^e
\]

with \( R_g^e = R_2(90 - \phi)R_3(\lambda) \)

\[ \Rightarrow \psi(Ex, Ey) \]

Fig: (Normalized) original and modified wavelet on the sphere
Spherical grid

Fig: Fibonacci grid (depth = 100 km)

- well suited for standard wavelets
- not enough nodes for ‘elliptic’ wavelets
Spherical grid

Fig: Fibonacci grid (depth = 100 km)

Fig: ‘scaled helix grid’
Example

- **GRACE-like orbit parameter**
- energy balance approach
- subtraction of a reference field
- ‘regional’ selection
- statistic in \([\text{m}^2/\text{s}^2]\):

<p>| | |</p>
<table>
<thead>
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<tbody>
<tr>
<td>MEAN</td>
<td>0.1191</td>
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<tr>
<td>MAX</td>
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<tr>
<td>MIN</td>
<td>−0.5464</td>
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<td>STD</td>
<td>0.2774</td>
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Synthesis

Fig: Synthesis by radial and ‘elliptic’ wavelets

<table>
<thead>
<tr>
<th></th>
<th>radial</th>
<th>‘elliptic’</th>
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<tr>
<td>nodes</td>
<td>623</td>
<td>598</td>
</tr>
<tr>
<td>$\text{cond}(A^\top A)$</td>
<td>1709</td>
<td>495</td>
</tr>
<tr>
<td>regularization</td>
<td>$3.58 \cdot 10^{-7}$</td>
<td>$2.56 \cdot 10^{-7}$</td>
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<tr>
<td>correlation [%]</td>
<td>0.94 (0.99)</td>
<td>0.93 (0.99)</td>
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Residuals in the inner zone

Fig: residuals after wavelet synthesis

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<th>[m²/s²]</th>
<th>synthesis</th>
<th>residuals</th>
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<tbody>
<tr>
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<tr>
<td>MAX</td>
<td>1.0101</td>
<td>1.0325</td>
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<td>MIN</td>
<td>−0.6260</td>
<td>−0.6138</td>
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<td>STD</td>
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<td>0.2769</td>
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Comparison

So far, similar quality for both kinds of wavelets

But for ‘elliptic’ wavelets
- smaller condition number and regularization parameter
- consideration of observation geometry
- improvements by grid modifications
construction of spherical base functions
- non-isotropic/directional dependent
- re-scaling in East-West direction with latitude
- in spatial domain
- easy/fast calculation
Summary

- construction of spherical base functions
  - non-isotropic/directional dependent
  - re-scaling in East-West direction with latitude
  - in spatial domain
  - easy/fast calculation

Open Questions

- scaling should depend on orbital parameters \((l,r)\)
- unchanged caps in higher/lower latitudes?
- analysis in other directions?
- special grid design?