

Lecture Notes  
Satellitengeodäsie (BSc)

# **Dynamic Satellite Geodesy**

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These are lecture notes in progress. Please contact me ([sneeuw@gis.uni-stuttgart.de](mailto:sneeuw@gis.uni-stuttgart.de)) for remarks, errors, suggestions, etc.

# Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction</b>   | <b>6</b>  |
| <b>2</b> | <b>The two-body problem</b>   | <b>7</b>  |
| 2.1      | Kepler's laws . . . . .   | 7         |
| 2.1.1    | First law: elliptical motion . . . . .                                  | 8         |
| 2.1.2    | Second law: area law . . . . .  | 10        |
| 2.1.3    | Third law . . . . .   | 11        |
| 2.2      | Further geometry . . . . .  | 13        |
| 2.2.1    | Three-dimensional orbit description . . . . .                           | 13        |
| 2.2.2    | Back to the orbital plane: anomalies . . . . .                          | 14        |
| 2.3      | Newton equations and conservation laws . . . . .                        | 16        |
| 2.3.1    | Conservation of energy . . . . .  | 17        |
| 2.3.2    | Conservation of angular momentum . . . . .                              | 17        |
| 2.3.3    | Conservation of orbit vector . . . . .                                  | 18        |
| 2.3.4    | Vis viva – living force . . . . .                                       | 19        |
| 2.4      | Further useful relations . . . . .                                      | 20        |
| 2.4.1    | Understanding Kepler . . . . .  | 20        |
| 2.4.2    | Partial derivatives $\nu \leftrightarrow E \leftrightarrow M$ . . . . . | 22        |
| 2.4.3    | Hohmann transfer orbit . . . . .  | 23        |
| 2.5      | Transformations Kepler $\longleftrightarrow$ Cartesian . . . . .        | 24        |
| 2.5.1    | Kepler $\longrightarrow$ Cartesian . . . . .                            | 24        |
| 2.5.2    | Cartesian $\longrightarrow$ Kepler . . . . .                            | 24        |
| <b>3</b> | <b>Equations of perturbed motion</b>                                    | <b>27</b> |
| 3.1      | Lagrange Planetary Equations . . . . .                                  | 27        |
| 3.1.1    | Earth oblateness . . . . .  | 29        |
| 3.2      | Canonical Equations . . . . .   | 31        |
| 3.2.1    | Newton equations . . . . .  | 31        |
| 3.2.2    | Delaunay . . . . .  | 32        |
| 3.2.3    | Hill . . . . .  | 32        |
| 3.3      | Gauss form . . . . .  | 32        |

|          |   |           |
|----------|---|-----------|
| <b>4</b> | <b>A viable alternative: Hill Equations</b>                 | <b>37</b> |
| 4.1      | Acceleration in a rotating reference frame . . . . .        | 37        |
| 4.2      | Hill equations . . . . .                                    | 40        |
| 4.3      | Solutions of the Hill equations . . . . .                   | 43        |
| 4.3.1    | The homogeneous solution . . . . .                          | 44        |
| 4.3.2    | The particular solution . . . . .                           | 45        |
| 4.3.3    | The complete solution . . . . .                             | 47        |
| 4.3.4    | The resonant solution . . . . .                             | 48        |
| <b>5</b> | <b>The gravitational potential and its representation</b>   | <b>51</b> |
| 5.1      | Representation on the sphere . . . . .                      | 51        |
| 5.2      | Representation in Kepler elements . . . . .                 | 53        |
| 5.3      | Lumped coefficient representation . . . . .                 | 58        |
| 5.4      | Pocket guide of dynamic satellite geodesy . . . . .         | 59        |
| 5.5      | Derivatives of the geopotential . . . . .                   | 59        |
| 5.5.1    | First derivatives: gravitational attraction . . . . .       | 60        |
| 5.5.2    | Second derivatives: the gravity gradient tensor . . . . .   | 63        |
| <b>6</b> | <b>Gravitational orbit perturbations</b>                    | <b>66</b> |
| 6.1      | The $J_2$ secular reference orbit . . . . .                 | 66        |
| 6.2      | Periodic gravity perturbations in linear approach . . . . . | 69        |
| 6.3      | The orbit perturbation spectrum . . . . .                   | 72        |
| <b>7</b> | <b>Modeling CHAMP, GRACE and GOCE observables</b>           | <b>76</b> |
| 7.1      | The Jacobi integral . . . . .                               | 76        |
| 7.2      | Range, range rate and range acceleration . . . . .          | 77        |
| 7.3      | Spaceborne gravimetry . . . . .                             | 79        |
| 7.4      | GRACE-type gradiometry . . . . .                            | 80        |
| 7.5      | GOCE gradiometry . . . . .                                  | 80        |
| <b>8</b> | <b>Perturbation theory</b>                                  | <b>81</b> |
| 8.1      | Classifications . . . . .                                   | 81        |
| 8.1.1    | Gravitationelle Störungen/Erdschwerefeld . . . . .          | 81        |
| 8.2      | $J_2$ -perturbations . . . . .                              | 82        |
| 8.2.1    | Qualitative assessment . . . . .                            | 82        |
| 8.2.2    | Kreiseffekte ← Qualitative Betrachtung . . . . .            | 82        |
| 8.2.3    | Quantitative assessment . . . . .                           | 83        |

|          |  |           |
|----------|--|-----------|
| <b>9</b> | <b>Non gravitational orbit perturbations</b> | <b>88</b> |
| 9.1      | Solar radiation pressure . . . . .           | 88        |
| 9.2      | Atmospheric drag . . . . .                   | 89        |

# 1 Introduction

Dynamic satellite geodesy is the application of celestial mechanics to geodesy. It aims in particular at describing satellite orbits under the influence of gravitational and non-gravitational forces. Conversely, if we know how orbit perturbations arise from gravity field disturbances, we have a tool for gravity field recovery from orbit analysis.

## 2 The two-body problem

The two-body problem is concerned with the motion of two gravitating masses,  $M$  and  $m$ , for instance planets around the Sun or satellites around the Earth. For convenience we consider  $M$  as the main attracting mass, and the orbiting mass  $m \ll M$ . This is not a mathematical necessity, though.

### 2.1 Kepler's laws

Kepler<sup>1</sup> was the first to give a proper mathematical description of (planetary) orbits. Dissatisfied with the mathematical trickery of the geocentric cosmology, necessary to explain astronomical observations of planetary motion, he was an early adopter of the Copernican heliocentric model. Although a mental breakthrough at the time, Kepler even went further.

Based on observations, most notably from the Danish astronomer Brahe<sup>2</sup>, Kepler empirically formulated three laws, providing a geometric-kinematical description of planetary motion. The first two laws were presented 1609 in his *Astronomia Nova* (The New Astronomy), the third one 1619 in *Harmonice Mundi* (Harmony of the World). They are:

- i) Planets move on an elliptical path around the sun, which occupies one of the focal points.
- ii) The line between sun and planet sweeps out equal areas in equal periods of time.
- iii) The ratio between the cube of the semi-major axis and the square of the revolution period is constant.

---

<sup>1</sup>Johannes Kepler (1571–1630). Born in Weil der Stadt, lived in Leonberg, studied at Tübingen University. Being unable to obtain a faculty position at Tübingen University, he became mathematics teacher in Graz. He later became research associate with Tycho Brahe in Prague and—after Brahe died—succeeded him as imperial mathematician.

<sup>2</sup>Tycho Brahe (1546–1626). This project website at the library of the ETH Zürich contains extensive information (in German).

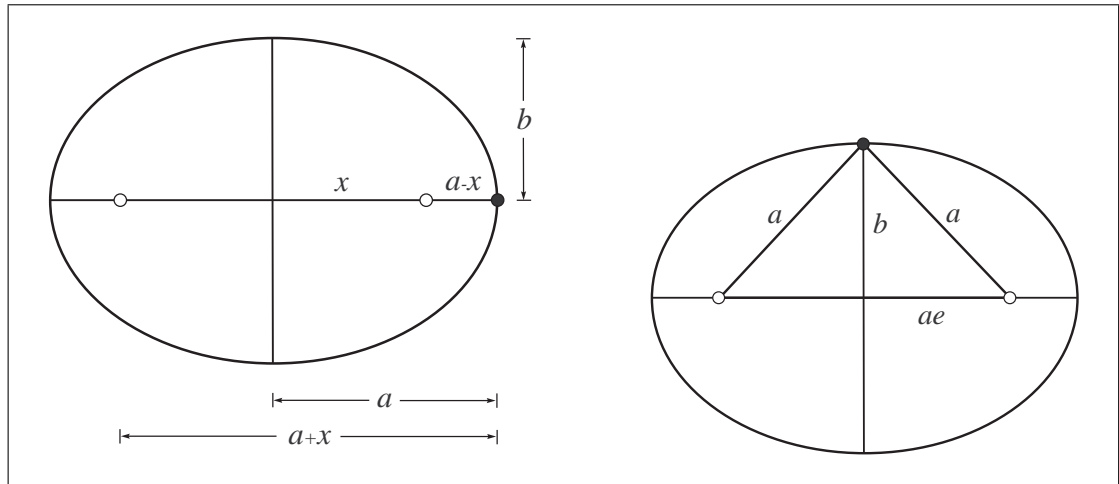
### 2.1.1 First law: elliptical motion

According to Kepler, planets move in ellipses around the sun. Although that was a daring statement already at a time when church dogma still prevailed scientific thought, Kepler even put the Sun outside the geometric centre of these ellipses. Instead he asserted that the Sun is at one of the foci.

#### Geometry

An ellipse is defined as the set of points whose sum of distances to both foci is constant. Inspection of fig. 2.1, in which we choose a point on the major axis (left panel), tells us that this sum must be  $(a + x) + (a - x) = 2a$ , the length of the major axis. The quantity  $a$  is called the *semi-major axis*.

lange Halbachse



**Figure 2.1:** Geometry of the Kepler ellipse in the orbital plane.

But then, for a point on the minor axis, see right panel, we have a symmetrical configuration. The distance from this point to each of the foci is  $a$ . The length  $b$  is called the *semi-minor axis*. Knowing both axes, we can express the distance to focus and centre of the ellipse. It is  $\sqrt{a^2 - b^2}$ . Usually it is expressed as a proportion  $e$  of the semi-major axis  $a$ :

kurze Halbachse

$$(ae)^2 + b^2 = a^2 \implies e^2 = \frac{a^2 - b^2}{a^2}, \text{ or } b = \sqrt{1 - e^2} a.$$

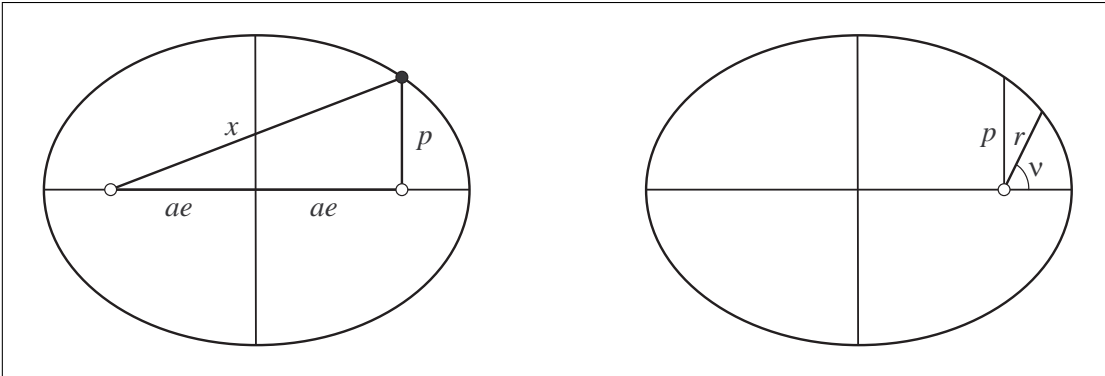
Exzentrizität

The proportionality factor  $e$  is called the *eccentricity*; the out-of-centre distance  $ae$  is known as the *linear eccentricity*.

From mathematics we know the polar equation of an ellipse:

$$r(\nu) = \frac{p}{1 + e \cos \nu}, \tag{2.1}$$





**Figure 2.2:** Parameters of the polar equation for the ellipse.

in which  $r$  is the radius,  $\nu$  the *true anomaly* and  $p$  the parameter of the ellipse. From the left panel of fig. 2.2 we are able to express  $p$  in terms of  $a$  and  $e$ . We can write down two equations:

wahre Anomalie

1. sum of sides:  $p + x = 2a$  or  $x^2 = 4a^2 - 4ap + p^2$
  2. Pythagoras:  $x^2 = p^2 + 4a^2e^2$
- eliminate  $x$ :  $p^2 + 4a^2e^2 = 4a^2 - 4ap + p^2$   
delete  $p^2$ :  $ae^2 = a - p$   
rewrite:  $p = a(1 - e^2)$ .

Knowing  $p$ , we recast the polar equation (2.1) into:

$$r(\nu) = \frac{a(1 - e^2)}{1 + e \cos \nu} \quad \text{or} \quad \frac{a(1 + e)(1 - e)}{1 + e \cos \nu} \quad (2.2)$$

**Exercise 2.1** Insert  $\nu = 0$  or  $180^\circ$  and check whether the outcome of (2.2) makes sense.

The orbital point closest to the mass-bearing focus is called *perihelion* in case of planetary motion around the Sun (Helios) or *perigee* for satellite motion around the Earth (Gaia). More general one can speak of *perifocus*. The farthest point, at  $\nu = 180^\circ$  is called, respectively, *aphelion*, *apogee* or *apofocus*. Since we are mostly discussing satellite motion, we will predominantly use perigee and apogee.

Perihel

Perigäum

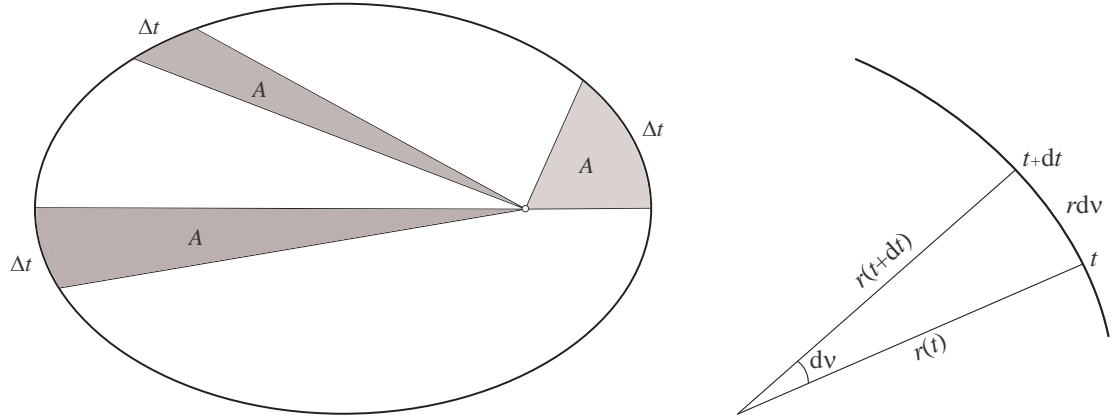
Aphel, Apogäum

**Remark 2.1 (circular orbit)** In case of zero eccentricity ( $e = 0$ ) the ellipse becomes a circle and  $a = b = p = r$

### 2.1.2 Second law: area law

Flächensatz

The line through focus and satellite (or planet) sweeps out equal areas  $A$  during equal intervals of time  $\Delta t$ . This is also known as Kepler's area law. From the left panel of fig. 2.3 it is seen that this effect is most extreme if a time interval around perigee is compared to one at apogee.



**Figure 2.3:** Kepler's area law (left) and infinitesimal area (right).

Bahnumlauf

As a consequence of Kepler's second law, the angular velocity  $\dot{\nu}$  must be variable during an orbital revolution: fast around perigee and slow around apogee.

The infinitesimal picture of this law looks as follows. In an infinitesimal amount of time  $dt$  the satellite travels an arc segment  $r d\nu$ . The infinitesimal, nearly triangular, reads  $dA = \frac{1}{2} r^2 d\nu$ . Therefore:

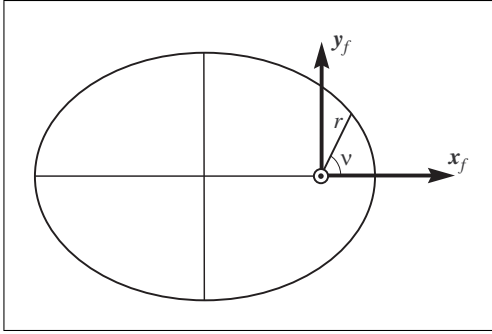
$$\begin{aligned} dA &= \frac{1}{2} r^2 d\nu \sim dt \\ \implies r^2 d\nu &= c dt \\ \implies r^2 \dot{\nu} &= c \end{aligned}$$

Drehimpuls

This sheds a different light on the area of Kepler's law. It is the quantity  $r^2 \dot{\nu}$  that is conserved. In a later section we will bring this in connection to the conservation of *angular momentum*. Here we can see already that, if we write  $v = r \dot{\nu}$  for linear velocity,  $rv$  is constant.

### Angular Momentum

Consider the perifocal coordinate system in fig. 2.4. In this frame the position and



**Figure 2.4:** Perifocal frame:  $x_f$  towards perigee,  $z_f$  perpendicular to orbital plane towards angular momentum, and  $y_f$  complementary in right-hand sense.

velocity vector read:

$$\mathbf{r}_f = \begin{pmatrix} r \cos \nu \\ r \sin \nu \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_f = \begin{pmatrix} \dot{r} \cos \nu - r\dot{\nu} \sin \nu \\ \dot{r} \sin \nu + r\dot{\nu} \cos \nu \\ 0 \end{pmatrix}.$$

The angular momentum vector, by its very definition, will be perpendicular to both and thus perpendicular to the orbital plane:

$$\mathbf{L}_f = \mathbf{r}_f \times \mathbf{v}_f = \begin{pmatrix} 0 \\ 0 \\ r\dot{r} \cos \nu \sin \nu + r^2 \cos^2 \nu \dot{\nu} - r\dot{r} \sin \nu \cos \nu + r^2 \sin^2 \nu \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r^2 \dot{\nu} \end{pmatrix}$$

### 2.1.3 Third law

Kepler's third law can be rephrased as

*The cubes of the semi-major axes of the orbits are proportional to the squares of the revolution periods.*

If we cast this law into mathematics, we obtain with proportionality factor  $c$ :

$$a^3 \sim T^2 \iff a^3 = cT^2.$$

The orbital period  $T$  is inversely related to the mean orbital angular velocity  $n$ :

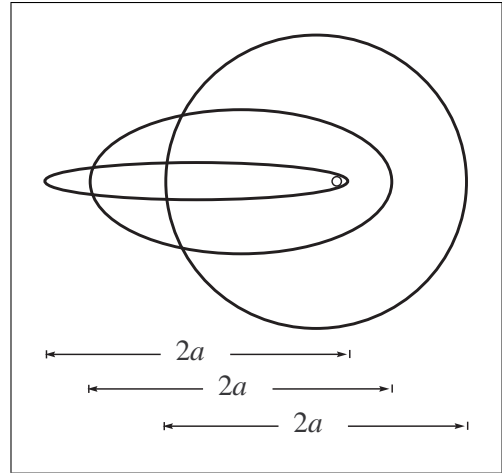
$$T = \frac{2\pi}{n}.$$

The angular velocity  $n$  is conventionally referred to as *mean motion*. We now obtain:

mittlere Bewegung

$$a^3 = c \frac{(2\pi)^2}{n^2} \implies n^2 a^3 = c(2\pi)^2.$$

**Figure 2.5:** The eccentricity  $e$  is not involved in Kepler’s third law. A circular orbit of radius  $a$  apparently has the same orbital revolution period as a highly eccentric (cigar-shaped) orbit of semi-major axis  $a$  as in fig. 2.5. Nevertheless, at  $e = 0$  one orbit has a length of  $2\pi a$  (the circumference of a sphere), whereas if the eccentricity approaches 1, one revolution approaches  $4a$  ( $2a$  forth plus  $2a$  back).



After Newton had developed his universal law of gravitation the seemingly arbitrary constant right hand side turned out to be more fundamental: the gravitational constant  $G$  times the mass  $M$  of the attracting body.

$$\boxed{n^2 a^3 = GM} \tag{2.3}$$

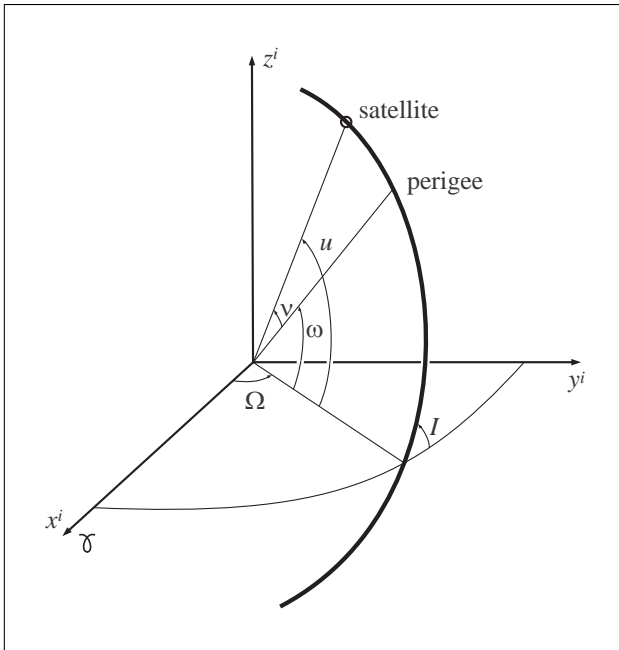
Although Kepler’s third law is intriguing, the particular combination of powers—a square and a cube—should not come as a surprise. Compare the situation of a circular orbit with angular velocity  $\omega$ . The centripetal force (per mass unit) is balanced by the gravitational attraction:

$$\omega^2 r = \frac{GM}{r^2} \implies \omega^2 r^3 = GM,$$

which is of the same form as Kepler’s third law.

Many orbits and orbital features can be calculated using (2.3). A few examples:

- geostationary orbit:  $n = \frac{2\pi}{1 \text{ day}} = \omega_d \implies a \approx 40\,000 \text{ km}$
- GPS:  $n = 2\omega_E \implies a = \dots$
- LEO:  $n \approx 15\omega_E$
- satellite at zero height:  $n \approx 16\omega_E \implies$  Schuler frequency



**Figure 2.6:** Three-dimensional geometry of the Kepler orbit.

## 2.2 Further geometry

### 2.2.1 Three-dimensional orbit description

The Kepler ellipse was defined in size by its semimajor axis  $a$  and in shape by its eccentricity  $e$ . The location of the satellite within the orbit was indicated by the true anomaly  $\nu$ . In three-dimensional space, though, we need two more parameters to indicate the orientation of the orbital plane, and again one more to orient the ellipse within this plane. In total we thus have 6 orbital elements or Kepler elements. The number 6 is equal to the the sum of 3 Cartesian position coordinates and 3 velocity components. Please refer to fig. 2.6.

The orbital plane is inclined with respect to the equator. The corresponding angle  $I$  is obviously called *inclination*. The intersection line between orbital plane and equator is the *nodal line*. The node, in which the satellite crosses the equator from South to North is the *ascending node*. The angle  $\Omega$  in inertial space from the *vernal equinox* (or  $x_i$ -axis) and ascending node is the *right ascension* of the ascending node. The *angle of perigee*  $\omega$  is counted from ascending node to perigee. The sum of angle of perigee and true anomaly is referred to as *argument of latitude*:  $u = \omega + \nu$ . This is a useful angle for circular and near-circular orbit for which the perigee is not or weakly defined.

Bahnneigung  
Knotenlinie  
steigender Knoten  
Frühlingspunkt  
Rektaszension  
Perigäumswinkel

We can classify the 6 Kepler elements as follows:

---

$a, e$  – size and shape of ellipse  
 $\Omega, I$  – orientation of orbital plane in space  
 $\omega, \nu$  – position within orbital plane

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Another classification is the following:

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$a, e, I$  – metric Kepler elements  
 $\Omega, \omega, \nu$  – angular Kepler elements

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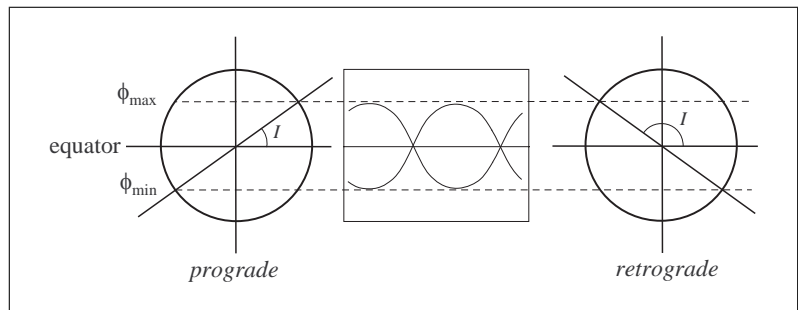
For the inclination we have in general  $I \in [0; 180)$ . Depending on the specific inclination (range) the orbits are known as:

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$I = 0^\circ$  – equatorial  
 $I < 90^\circ$  – prograde  
 $I = 90^\circ$  – polar  
 $I > 90^\circ$  – retrograde

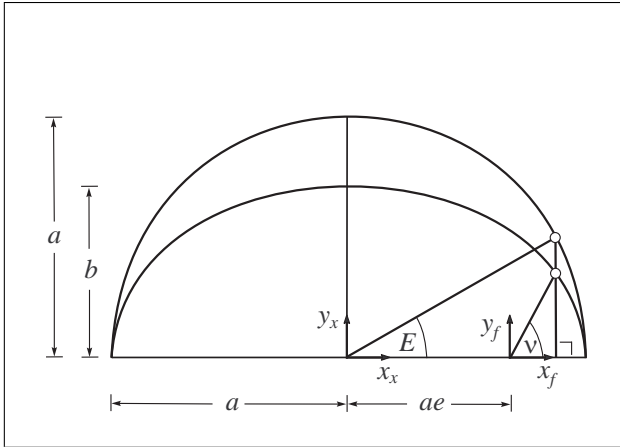
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**Figure 2.7:** The inclination determines the maximum and minimum latitude that ground-tracks can attain:  $\phi_{\min}, \phi_{\max}$ .



### 2.2.2 Back to the orbital plane: anomalies

From Kepler's area law it was clear that  $\nu$  is not uniform in time. In order to describe the time evolution more explicitly Kepler introduces two more angles: eccentric anomaly  $E$  and mean anomaly  $M$ . The latter will be uniform in time, in the sense that we will be able to write  $M = n(t - t_0)$  later on.



**Figure 2.8:** Definition of eccentric anomaly  $E$  from true anomaly  $\nu$ . The geometric construction is similar to the definition of a reduced latitude from a geodetic latitude.

**eccentric anomaly** Consider fig. 2.8 with the perifocal  $f$ -frame and the eccentric  $x$ -frame. In the perifocal frame the position vector reads:

$$\mathbf{r}_f(r, \nu) = \begin{pmatrix} r \cos \nu \\ r \sin \nu \\ 0 \end{pmatrix} \quad (2.4)$$

Using the position vector in the eccentric frame we derive:

$$\mathbf{r}_x(a, E) = \begin{pmatrix} a \cos E \\ b \sin E \\ 0 \end{pmatrix} \implies \mathbf{r}_f(a, E) = \begin{pmatrix} a \cos E - ae \\ a\sqrt{1-e^2} \sin E \end{pmatrix}. \quad (2.5)$$

After some manipulation we find:

$$r = a(1 - e \cos E). \quad (2.6)$$

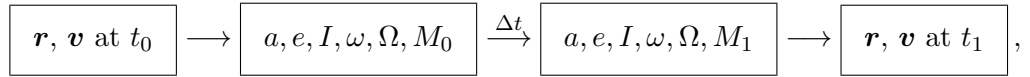
**mean anomaly** Neither  $\nu$  nor  $E$  is uniform (linear in time). Kepler therefore defined the mean anomaly  $M$ . The following equation is usually referred to as the *Kepler equation*:

$$\boxed{M = E - e \sin E}. \quad (2.7)$$

The mean anomaly is a fictitious angle. It cannot be drawn in fig. 2.8. It can only be calculated from  $E$ . But it evolves linearly in time. Thus one can write:

$$M = n(t - t_0),$$

in which  $t_0$  stands for the time of perigee passage, where  $\nu = E = M = 0$ . This allows us to calculate the orbit evolution over time, say from  $t_0$  to  $t_1$  by the following scheme:



in which the time step  $\Delta t$  stands more explicitly for  $M_1 = M_0 + n(t_1 - t_0)$ . The (Cartesian) position and velocity vectors at  $t_0$  are known as the *initial state*. In summary, if one wants to know the orbital position and velocity as a function of time, one should transform the initial state into Kepler elements. In the Kepler element domain, only the mean anomaly  $M$  changes over time. To be precise, it changes linearly with time. After the time step, the Kepler elements need to be transformed back to position and velocity again

**Reverse Kepler equation** For the reverse transformation of the Kepler equation (2.7) an iteration is required:

$$\begin{aligned} M &\longrightarrow E : \text{iterate } E_{i+1} = e \sin E_i + M \\ E_0 &= 0 \\ E_1 &= M \\ E_2 &= e \sin M + M \\ E_3 &= \dots \end{aligned}$$

### 2.3 Newton equations and conservation laws

Kepler's laws provide a geometric and kinematic picture of orbital motion. Although the area law hints at angular momentum conservation already and the third law at gravitation, the concept of forces was unknown to Kepler. A dynamic description of the Kepler orbit had to wait for Newton. Moreover, Kepler derived his laws empirically.

In this section we will take Newton's equations of motion for the two-body problem:

$$\ddot{\mathbf{r}} = \nabla \frac{GM}{r} = -\frac{GM}{r^3} \mathbf{r}, \text{ or } \boxed{\ddot{\mathbf{r}} + \frac{GM}{r^3} \mathbf{r} = 0}, \quad (2.8)$$

and apply three tricks to it:

- i) scalar multiplication with velocity:  $\dot{\mathbf{r}} \cdot \dots$ ,
- ii) vectorial multiplication with velocity:  $\dot{\mathbf{r}} \times \dots$ ,



iii) vectorial multiplication with angular momentum:  $\mathbf{L} \times \dots$

After a subsequent time integration we will end up with fundamental conservation laws and, eventually, with the Kepler orbit. Thus, at the end of this section we will have achieved a dynamical description of the Kepler orbit, based on a physical principle.

### 2.3.1 Conservation of energy

Trick 1: “ $\dot{\mathbf{r}}$ · Newton”.

**Remark 2.2** If  $\dot{\mathbf{r}} = \mathbf{v}$  and  $r$  and  $v$  are respectively the length of  $\mathbf{r}$  and  $\mathbf{v}$ , then  $\dot{r} \neq v$ ! Instead, the scalar radial velocity is only the projection of the velocity vector on the radial direction:  $\dot{r} = \dot{\mathbf{r}} \cdot \mathbf{r}/r$ .

$$\begin{aligned} \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \frac{GM}{r^3} \mathbf{r} &= 0 \\ \iff \mathbf{v} \cdot \dot{\mathbf{v}} + \frac{GM}{r^3} \dot{\mathbf{r}} \cdot \mathbf{r} &= 0 \\ \iff \mathbf{v} \cdot \dot{\mathbf{v}} + \frac{GM}{r^2} \dot{r} &= 0, \text{ (because } \dot{r}r = \dot{\mathbf{r}} \cdot \mathbf{r}\text{)} \\ \iff \frac{1}{2} \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) - \frac{d}{dt} \left( \frac{GM}{r} \right) &= 0 \\ \iff \frac{d}{dt} \left( \frac{1}{2} v^2 - \frac{GM}{r} \right) &= 0 \\ \implies \frac{1}{2} v^2 - \frac{GM}{r} &= c. \end{aligned}$$

This demonstrates that the sum of kinetic and potential energy is constant:  $c = E$ . Later we will evaluate the exact amount of energy using the vis-viva equation.

### 2.3.2 Conservation of angular momentum

Trick 2: “ $\mathbf{r} \times$  Newton”.

$$\begin{aligned} \mathbf{r} \times \ddot{\mathbf{r}} + \frac{GM}{r^3} \underbrace{\mathbf{r} \times \mathbf{r}}_{=0} &= \mathbf{0} \\ \iff \mathbf{r} \times \ddot{\mathbf{r}} &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) &= \underbrace{\dot{\mathbf{r}} \times \dot{\mathbf{r}}}_{=0} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}, \text{ (lucky guess)} \\ \Rightarrow \mathbf{r} \times \dot{\mathbf{r}} &= \mathbf{c} \end{aligned}$$

This demonstrates that the angular momentum  $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$  is constant:  $\mathbf{c} = \mathbf{L}$ . We have more or less reproduced Kepler's area law from Newton's equation. Note, however, that we have achieved here conservation of the 3D angular momentum vector. Not only are the areas equal over equal times (one dimension), but also is the orbital plane constant in inertial space (two dimensions). The latter will lead to  $\Omega$  and  $I$ .

### 2.3.3 Conservation of orbit vector

Trick 3: "Newton  $\times$   $\mathbf{L}$ ".

$$\begin{aligned} \ddot{\mathbf{r}} \times \mathbf{L} + \frac{GM}{r^3} \mathbf{r} \times \mathbf{L} &= \mathbf{0} \\ \Leftrightarrow \underbrace{\ddot{\mathbf{r}} \times \mathbf{L}}_{\text{LHS}} &= \underbrace{\frac{GM}{r^3} \mathbf{L} \times \mathbf{r}}_{\text{RHS}} \\ \text{LHS : } \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{L}) &= \ddot{\mathbf{r}} \times \mathbf{L} + \mathbf{r} \times \underbrace{\dot{\mathbf{L}}}_{=0} \\ \text{RHS : } \frac{GM}{r^3} \mathbf{r} \times \mathbf{L} &= \frac{GM}{r^3} (\mathbf{r} \times \dot{\mathbf{r}}) \times \mathbf{r} \\ &= \frac{GM}{r^3} [(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \\ &= \frac{GM}{r} \dot{\mathbf{r}} - \frac{GM}{r^2} \dot{r} \mathbf{r} \\ GM \frac{d}{dt} \frac{\mathbf{r}}{r} &= \frac{GM}{r} \dot{\mathbf{r}} - \frac{GM}{r^2} \dot{r} \mathbf{r}, \text{ (lucky guess)} \\ \text{LHS = RHS : } \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{L}) &= GM \frac{d}{dt} \frac{\mathbf{r}}{r} \\ \Rightarrow \dot{\mathbf{r}} \times \mathbf{L} &= \frac{GM}{r} \mathbf{r} + \mathbf{B} \end{aligned}$$

The vector  $\mathbf{B}$  is a constant of the integration. It is a quantity that is conserved in the two-body problem. It is known as Runge-Lenz vector or Laplace vector. The above derivation shows that  $\mathbf{B}$  is a linear combination of  $\dot{\mathbf{r}} \times \mathbf{L}$  and  $\mathbf{r}$ . Therefore  $\mathbf{B}$  must lie in the orbital plane.

The last equation can be written in a different form if we perform scalar multiplication with the position vector:  $\mathbf{r} \cdot \dots$

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{L}) = \frac{GM}{r} \mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{B} \quad (2.9)$$

Under cyclic permutation the left-hand side is equal to  $\mathbf{L} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{L}\mathbf{L} = L^2$ .

$$\implies L^2 = GMr + r|\mathbf{B}| \cos \alpha$$

$$\implies r = \frac{\frac{L^2}{GM}}{1 + \frac{|\mathbf{B}|}{GM} \cos \alpha}$$

If we now identify the following quantities:

$$\alpha := \nu \quad , \quad \frac{L^2}{GM} := p \quad , \quad \frac{|\mathbf{B}|}{GM} := e \quad ,$$

then we obtain the polar equation of the ellipse (2.1) again:

$$r(\nu) = \frac{p}{1 + e \cos \nu} .$$

At the same time we have learnt that the Laplace vector  $\mathbf{B}$  points towards perigee.

**Remark 2.3** *Effectively we have now solved the Kepler problem using Newton's equation of motion. We have implicitly obtained Kepler's laws.*

### 2.3.4 Vis viva – living force

It was demonstrated that the total energy  $E$  is conserved:

$$\begin{aligned} \frac{1}{2}v^2 - \frac{GM}{r} &= E \\ T + V &= E \end{aligned}$$

Question: how much constant energy?

$$\begin{aligned} L &= |\mathbf{L}| = rv = r_{\text{apo}}v_{\text{apo}} = r_{\text{per}}v_{\text{per}} \\ \implies E &= \frac{L^2}{2r^2} - \frac{GM}{r} \stackrel{\text{e.g.}}{=} \frac{L^2}{2r_{\text{per}}^2} - \frac{GM}{r_{\text{per}}} \end{aligned}$$

$$\begin{aligned}
 p &= a(1 - e^2) = \frac{L^2}{GM} \\
 \implies E &= \frac{GMa(1 - e^2)}{2a^2(1 - e^2)} - \frac{GM}{a(1 - e^2)} \\
 &= \frac{1}{2}GM \frac{1 + e}{a(1 - e)} - \frac{GM}{a(1 - e^2)} \\
 &= \frac{GM}{a(1 - e)} \left[ \frac{1}{2}(1 + e) - 1 \right] \\
 &= -\frac{GM}{2a}
 \end{aligned}$$

$$\boxed{\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}}. \quad (2.10)$$

**Remark 2.4 (Hohmann transfer orbits)** *Energy level only depends on  $a$ . Revolution period also. Hohmann transfer brings a satellite from one circular orbit into another (higher) circular orbit by two so-called  $\Delta v$ -thrusts. The first thrust—a short engine burn at perigee passage—brings the satellite into an elliptical transfer orbit. A second boost at apogee circularizes the orbit.*

## 2.4 Further useful relations

### 2.4.1 Understanding Kepler

**Perifocal;  $r, \nu$**

$$\mathbf{r}_f = \begin{pmatrix} r \cos \nu \\ r \sin \nu \\ 0 \end{pmatrix} \quad \dot{\mathbf{r}}_f = \begin{pmatrix} \dot{r} \cos \nu - r\dot{\nu} \sin \nu \\ \dot{r} \sin \nu + r\dot{\nu} \cos \nu \\ 0 \end{pmatrix}$$

$$L\mathbf{H}_f = \mathbf{r}_f \times \dot{\mathbf{r}}_f = \begin{pmatrix} 0 \\ 0 \\ r^2\dot{\nu} \end{pmatrix} \quad \mathbf{r}_f \cdot \dot{\mathbf{r}}_f = r\dot{r}$$

**Eccentric;  $a, E$**

$$\mathbf{r}_x = \begin{pmatrix} a \cos E \\ b \sin E \\ 0 \end{pmatrix} \quad \dot{\mathbf{r}}_x = \begin{pmatrix} -a\dot{E} \sin E \\ b\dot{E} \cos E \\ 0 \end{pmatrix}$$

**Perifocal;  $a, E$**

$$\mathbf{r}_f = \begin{pmatrix} a \cos E - ae \\ a\sqrt{1-e^2} \sin E \\ 0 \end{pmatrix} \quad \dot{\mathbf{r}}_f = \begin{pmatrix} -a\dot{E} \sin E \\ a\sqrt{1-e^2} \dot{E} \cos E \\ 0 \end{pmatrix}$$

$$\begin{aligned} L &= |\mathbf{r} \times \dot{\mathbf{r}}| = L_{f=3} \\ &= a^2 \cos^2 E \sqrt{1-e^2} \dot{E} - a^2 e \sqrt{1-e^2} \dot{E} \cos E + a^2 \sqrt{1-e^2} \dot{E} \sin^2 E \\ &= a^2 \sqrt{1-e^2} \dot{E} - a^2 e \sqrt{1-e^2} \dot{E} \cos E \\ &= a^2 \sqrt{1-e^2} \dot{E} (1 - e \cos E) \end{aligned}$$

$$\begin{aligned} \text{also: } L &= \sqrt{GMa(1-e^2)} \\ \implies \sqrt{GMa(1-e^2)} &= a^2 \sqrt{1-e^2} \dot{E} (1 - e \cos E) \\ \iff \sqrt{\frac{GM}{a^3}} &= \dot{E} (1 - e \cos E) = n \end{aligned}$$

→ integrates to  $M = E - e \sin E$

→ explains Kepler's 3rd

$$\begin{aligned} \text{Now: } \mathbf{r} \cdot \dot{\mathbf{r}} &= -a^2 \cos E \sin E \dot{E} + a^2 e \dot{E} \sin E + a^2 (1-e^2) \dot{E} \sin E \cos E \\ &= a^2 e \dot{E} \sin E - a^2 e^2 \dot{E} \sin E \cos E \\ &= a^2 e \dot{E} \sin E (1 - e \cos E) \\ &= a^2 e n \sin E \end{aligned}$$

Also  $\mathbf{r} \cdot \dot{\mathbf{r}} = r \dot{r}$

$$\text{Now: } r = \sqrt{a^2 \cos^2 E - 2a^2 e \cos E + a^2 e^2 + a^2 (1-e^2) \sin^2 E}$$

$$\begin{aligned}
 &= \sqrt{a^2 - a^2 e^2 \sin^2 E - 2a^2 e \cos E + a^2 e^2} \\
 &= \sqrt{a^2 + a^2 e^2 \cos^2 E - 2a^2 e \cos E} \\
 &= a\sqrt{1 - 2e \cos E + e^2 \cos^2 E} \\
 &= a(1 - e \cos E)
 \end{aligned}$$

### 2.4.2 Partial derivatives $\nu \leftrightarrow E \leftrightarrow M$

Goal:

$$\frac{\partial \nu}{\partial M} = \frac{\partial \nu}{\partial E} \frac{\partial E}{\partial M}$$

The first part at the right side is difficult. We need to get back to the expression of the radial distance, both in terms of true anomaly  $\nu$  and of eccentric anomaly  $E$ .

$$\begin{aligned}
 r(\nu) &= \frac{a(1 - e^2)}{1 + e \cos \nu} \Rightarrow \frac{\partial r}{\partial \nu} = \frac{a(1 - e^2)}{(1 + e \cos \nu)^2} e \sin \nu = \frac{r^2 e \sin \nu}{a(1 - e^2)} \\
 r(E) &= a(1 - e \cos E) \Rightarrow \frac{\partial r}{\partial E} = ae \sin E
 \end{aligned}$$

Thus, we get:

$$\frac{\partial \nu}{\partial E} = \frac{\partial \nu}{\partial r} \frac{\partial r}{\partial E} = \frac{a(1 - e^2)}{r^2 \sin \nu} a \sin E$$

Remember that the  $y$ -coordinate in the perifocal frame can either be expressed as  $y_f = r \sin \nu$  or as  $y_f = b \sin E$ . Therefore, we end up with

$$\frac{\partial \nu}{\partial E} = \frac{a^2(1 - e^2)}{rb} = \frac{b^2}{rb} = \frac{b}{r}$$

The second part at the right side of the equation above is easily obtained from Kepler's equation:

$$M = E - e \sin E \Rightarrow \frac{\partial M}{\partial E} = 1 - e \cos E = \frac{r}{a}$$

Combining all information, we get:

$$\frac{\partial \nu}{\partial M} = \frac{\partial \nu}{\partial E} \frac{\partial E}{\partial M} = \frac{ab}{r^2}$$

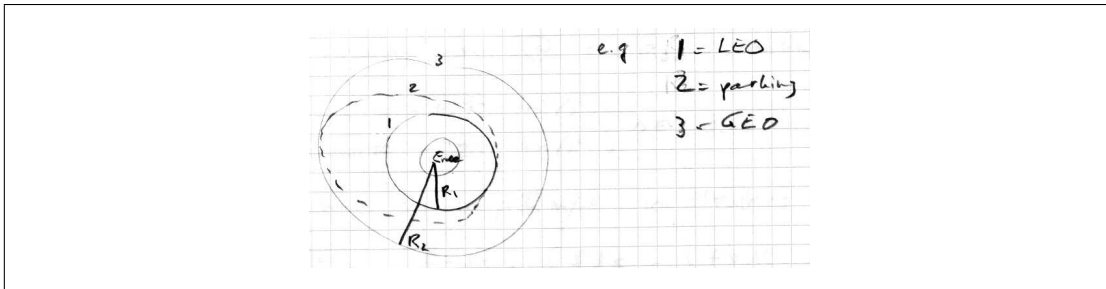


Figure 2.9

### 2.4.3 Hohmann transfer orbit

$$\begin{aligned} \text{Orbit 1: } a_1 &= R_1 & E_1 &= -\frac{GM}{2R_1} \\ \text{Orbit 2: } a_2 &= R_1 + R_2 & E_2 &= -\frac{GM}{2(R_1 + R_2)} \\ \text{Orbit 3: } a_3 &= R_2 & E_3 &= -\frac{GM}{2R_2} \end{aligned}$$

vis viva

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a} \quad (= E)$$

$$v^2 = 2\frac{GM}{r} - \frac{GM}{a}$$

$$v = \sqrt{GM \left( \frac{2}{r} - \frac{1}{a} \right)}$$

$$v_1 = \sqrt{GM \left( \frac{2}{R_1} - \frac{1}{R_1} \right)} = \sqrt{\frac{GM}{R_1}}$$

$$v_2(r) = \sqrt{GM \left( \frac{2}{r} - \frac{1}{R_1 + R_2} \right)} \quad \begin{aligned} r = R_1 &: \sqrt{GM \left( \frac{2}{R_1} - \frac{1}{R_1 + R_2} \right)} \\ r = R_2 &: \sqrt{GM \left( \frac{2}{R_2} - \frac{1}{R_1 + R_2} \right)} \end{aligned}$$

$$v_3 = \sqrt{\frac{GM}{R_2}}$$

## 2.5 Transformations Kepler $\longleftrightarrow$ Cartesian

### 2.5.1 Kepler $\longrightarrow$ Cartesian

**Problem:** Given 6 Kepler elements  $(a, e, I, \omega, \Omega, M)$ , find the corresponding inertial position  $\mathbf{r}_i$  and velocity  $\dot{\mathbf{r}}_i$ .

**Solution:** First get the eccentric anomaly  $E$  from the mean anomaly  $M$  by iteratively solving Kepler's equation:

$$E - e \sin E = M \Rightarrow E_{i+1} = e \sin E_i + M, \text{ with starting value } E_0 = M \quad (2.11)$$

Next, get the position and the velocity in the perifocal  $f$ -frame, which has its  $z$ -axis perpendicular to the orbital plane and its  $x$ -axis pointing to the perigee:

$$\mathbf{r}_f = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1 - e^2} \sin E \\ 0 \end{pmatrix}, \quad \dot{\mathbf{r}}_f = \frac{na}{1 - e \cos E} \begin{pmatrix} -\sin E \\ \sqrt{1 - e^2} \cos E \\ 0 \end{pmatrix} \quad (2.12)$$

In case the true anomaly  $\nu$  is given in the original problem instead of the mean anomaly  $M$ , the vectors  $\mathbf{r}_f$  and  $\dot{\mathbf{r}}_f$  are obtained by:

$$\mathbf{r}_f = \begin{pmatrix} r \cos \nu \\ r \sin \nu \\ 0 \end{pmatrix}, \quad \dot{\mathbf{r}}_f = \frac{na}{\sqrt{1 - e^2}} \begin{pmatrix} -\sin \nu \\ e + \cos \nu \\ 0 \end{pmatrix} \quad (2.13)$$

with

$$r = \frac{a(1 - e^2)}{1 + e \cos \nu} \quad (2.14)$$

The transformation from inertial  $i$ - frame to the perifocal  $f$ -frame is performed by the rotation sequence  $R_3(\omega)R_1(I)R_3(\Omega)$ . So, vice versa, the inertial position and velocity are obtained by the reverse transformations:

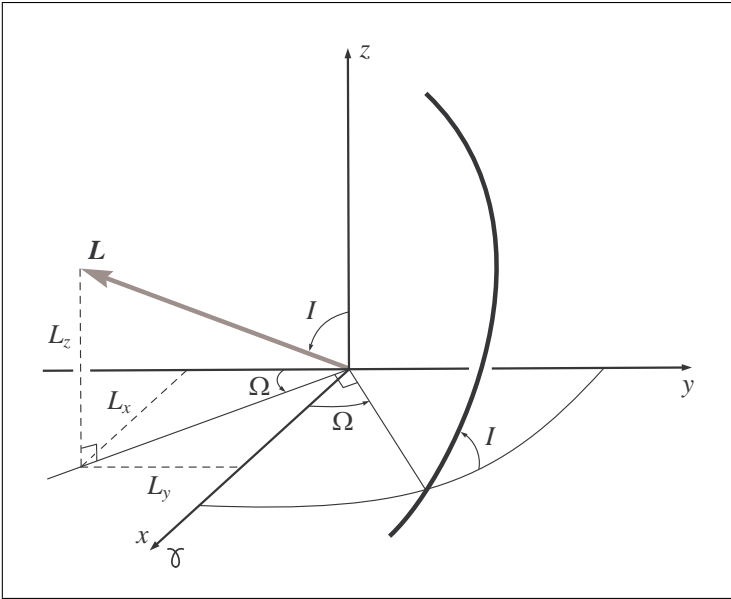
$$\mathbf{r}_i = R_3(-\Omega)R_1(-I)R_3(-\omega)\mathbf{r}_f \quad (2.15)$$

$$\dot{\mathbf{r}}_i = R_3(-\Omega)R_1(-I)R_3(-\omega)\dot{\mathbf{r}}_f \quad (2.16)$$

### 2.5.2 Cartesian $\longrightarrow$ Kepler

**Problem:** Given a satellite's inertial position  $\mathbf{r}_i$  and velocity  $\dot{\mathbf{r}}_i$ , find the corresponding Kepler elements  $(a, e, I, \omega, \Omega, M)$ .





**Figure 2.10:** The angular momentum vector  $\mathbf{L}$  defines the orientation of the orbital plane in terms of  $\Omega$  and  $I$ .

**Solution:** The angular momentum vector per unit mass is normal to the orbital plane. It defines the inclination  $I$  and right ascension of the ascending node  $\Omega$ :

$$\mathbf{L}_i = \mathbf{r}_i \times \dot{\mathbf{r}}_i \quad (2.17)$$

$$\tan \Omega = \frac{L_1}{-L_2} \quad (2.18)$$

$$\tan I = \frac{\sqrt{L_1^2 + L_2^2}}{L_3} \quad (2.19)$$

Rotate  $\mathbf{r}_i$  into the orbital plane now and derive the argument of latitude  $u$ :

$$\mathbf{p} = R_1(I)R_3(\Omega)\mathbf{r}_i \quad (2.20)$$

$$\tan u = \tan(\omega + \nu) = \frac{p_2}{p_1} \quad (2.21)$$

The semi-major axis  $a$  comes from the vis-viva equation and requires the scalar velocity  $v = |\dot{\mathbf{r}}|$ . The eccentricity  $e$  comes from the description of the Laplace-vector and needs the scalar angular momentum  $L = |\mathbf{L}|$ :

$$T - V = \frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{2a} \quad (2.22)$$

$$a = \frac{GM r}{2GM - rv^2} \quad (2.23)$$

$$e = \sqrt{1 - \frac{L^2}{GM a}} \quad (2.24)$$

In order to extract the eccentric anomaly  $E$ , we need to know the radial velocity first:

$$\dot{r} = \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r} \quad (2.25)$$

$$\cos E = \frac{a - r}{ae} \quad (2.26)$$

$$\sin E = \frac{r\dot{r}}{e\sqrt{GM a}} \quad (2.27)$$

The true anomaly is obtained from the eccentric one:

$$\tan \nu = \frac{\sqrt{1 - e^2} \sin E}{\cos E - e} \quad (2.28)$$

Subtracting  $\nu$  from the argument of latitude  $u$  yields the argument of perigee  $\omega$ . Finally, Kepler's equation provides the mean anomaly:

$$E - e \sin E = M \quad (2.29)$$

# 3 Equations of perturbed motion

## 3.1 Lagrange Planetary Equations

### Crash course LPE

Question:

$$\left. \begin{aligned} \dot{\mathbf{r}} &= \frac{\partial F}{\partial \mathbf{v}} \\ \dot{\mathbf{v}} &= -\frac{\partial F}{\partial \mathbf{r}} \end{aligned} \right\} \implies \dot{\mathbf{s}} = ?$$

$$\mathbf{s} = (a \ e \ I \ \Omega \ \omega \ M)^\top$$

$$\left\{ \begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{\partial \mathbf{r}}{\partial \mathbf{s}} \frac{d\mathbf{s}}{dt} \implies \dot{\mathbf{r}} = \mathbf{A} \dot{\mathbf{s}} = \frac{\partial T}{\partial \mathbf{v}} \\ \frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial \mathbf{s}} \frac{d\mathbf{s}}{dt} \implies \dot{\mathbf{v}} = \dot{\mathbf{A}} \dot{\mathbf{s}} = \frac{\partial V}{\partial \mathbf{r}} \end{aligned} \right.$$

Trick:

$$\left\{ \begin{aligned} \dot{\mathbf{A}}^\top \dot{\mathbf{r}} &= \dot{\mathbf{A}}^\top \mathbf{A} \dot{\mathbf{s}} = \dot{\mathbf{A}}^\top \frac{\partial T}{\partial \mathbf{v}} \\ \mathbf{A}^\top \dot{\mathbf{v}} &= \mathbf{A}^\top \dot{\mathbf{A}} \dot{\mathbf{s}} = \mathbf{A}^\top \frac{\partial V}{\partial \mathbf{r}} \end{aligned} \right.$$

$$\begin{aligned} (\mathbf{A}^\top \dot{\mathbf{A}} - \dot{\mathbf{A}}^\top \mathbf{A}) \dot{\mathbf{s}} &= \mathbf{A}^\top \frac{\partial V}{\partial \mathbf{r}} - \dot{\mathbf{A}}^\top \frac{\partial T}{\partial \mathbf{v}} \\ &= - \left( \dot{\mathbf{A}}^\top \frac{\partial F}{\partial \mathbf{v}} + \mathbf{A}^\top \frac{\partial F}{\partial \mathbf{r}} \right) \end{aligned}$$

$$L \dot{\mathbf{s}} = -\frac{\partial F}{\partial \mathbf{s}} \quad L = \text{Lagrangeklammern}$$

mit  $\mathbf{s} = (a \ e \ I \ \Omega \ \omega \ M)^\top \implies$  Lagrange Planetary Equation

### Eigenschaften $L$

- schief-symmetrisch:

$$L^\top = -L \implies 15 \text{ unabhangige Elemente}$$

- Zeitinvariant:

$$\begin{aligned}\dot{L} &= A^\top \ddot{A} - \ddot{A}^\top A = 0 \\ \ddot{A}_i k &= \frac{\partial \dot{v}_i}{\partial \mathbf{s}_k} = \frac{\partial^2 V}{\partial r_i \partial L_k} \\ (A^\top \ddot{A})_l k &= \sum_i \frac{\partial r_i}{\partial L_l} \frac{\partial^2 V}{\partial r_i \partial L_k} = \frac{\partial^2 V}{\partial L_l \partial L_k} = \text{symmetrisch} \\ &\implies \text{evaluieren z. B. im Perigaeum}\end{aligned}$$

### LPE

$$\dot{\mathbf{s}} = -S^{-1} \frac{\partial F}{\partial \mathbf{s}} \quad S^{-1} = \text{Poissonklammern}$$

- Bewegungsgleichungen in  $\mathbf{s}$ , z. B. Keplerelemente
- 6 GDGL 1. Ordnung
- nicht-linear

The equations of motion with disturbing potential<sup>1</sup>  $R$  in Cartesian coordinates are:

$$\ddot{\mathbf{r}} = \nabla \frac{GM}{r} + \nabla R \quad (3.1)$$

After transforming position  $\mathbf{r}$  and velocity  $\dot{\mathbf{r}}$  into Kepler elements, the equations of motion are called the Lagrange Planetary Equations:

$$\dot{a} = \frac{2}{na} \frac{\partial R}{\partial M} \quad (3.2a)$$

$$\dot{e} = \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial \omega} \quad (3.2b)$$

$$\dot{I} = \frac{\cos I}{na^2 \sqrt{1-e^2} \sin I} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1-e^2} \sin I} \frac{\partial R}{\partial \Omega} \quad (3.2c)$$

$$\dot{\omega} = -\frac{\cos I}{na^2 \sqrt{1-e^2} \sin I} \frac{\partial R}{\partial I} + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} \quad (3.2d)$$

$$\dot{\Omega} = \frac{1}{na^2 \sqrt{1-e^2} \sin I} \frac{\partial R}{\partial I} \quad (3.2e)$$

$$\dot{M} = n - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} \quad (3.2f)$$

---

<sup>1</sup>This implies that only gravitational forces can be treated in the following. For dissipative forces, the Gauss form of the equations of motion should be used.

$$\begin{pmatrix} a \\ e \\ I \\ \Omega \\ \omega \\ M \end{pmatrix} = \left( \begin{array}{c|ccc} & 0 & 0 & \frac{2}{na} \\ & 0 & -\frac{\sqrt{1-e^2}}{na^2e} & \frac{1-e^2}{na^2e} \\ & -\frac{1}{nab \sin I} & \frac{\cot I}{nab} & 0 \\ \hline \text{a-symm (?)} & & & 0 \end{array} \right) \begin{pmatrix} \frac{\partial F}{\partial a} \\ \frac{\partial F}{\partial e} \\ \frac{\partial F}{\partial I} \\ \frac{\partial F}{\partial \Omega} \\ \frac{\partial F}{\partial \omega} \\ \frac{\partial F}{\partial M} \end{pmatrix}$$

### 3.1.1 Earth oblateness

$C_{20}$  – Beispiel:

$$\begin{aligned} F &= T - V \\ &= \frac{1}{2}v^2 - \frac{GM}{r} - R_{2,0} \\ &= -\frac{GM}{20} - \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{20} P_{2,0}(\sin \phi) \\ \implies R_{2,0} &= \frac{1}{2} \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{20} (3 \sin^2 u \sin^2 I - 1) \end{aligned}$$

z. B.

$$\begin{aligned} \dot{\Omega} &= \frac{1}{nab \sin I} \frac{\partial R_{2,0}}{\partial I} \\ &= \frac{1}{2} \frac{1}{nab \sin I} \frac{n^2 a^3}{R} \left(\frac{R}{r}\right)^3 C_{20} \cdot 3 \underbrace{\sin^2 u}_{\frac{1}{2}(1-\cos(2u))} \cos I \sin I \\ &= \frac{3}{2} \frac{a^2}{br} \left(\frac{R}{r}\right)^2 C_{20} \cos I (1 - \cos(2u)) \\ \xrightarrow{\text{secular}} \dot{\Omega} &= \frac{3}{2} n \frac{C_{20}}{(1-e^2)^2} \left(\frac{R}{r}\right)^2 \cos I \end{aligned}$$

Beispiel:  $h = 750\text{km}$ ,  $e \approx 0$ ,  $C_{20} = -10^{-3}$

Here we are only interested in the secular effect of the Earth's flattening, which is parameterized by the unnormalized coefficient  $C_{20} = -1.08263 \cdot 10^{-3}$ . The equations of motion reduce to:

$$\dot{a} = 0 \tag{3.3a}$$

$$\dot{e} = 0 \tag{3.3b}$$

$$\dot{I} = 0 \quad (3.3c)$$

$$\dot{\omega} = \frac{3nC_{20}a_E^2}{4(1-e^2)^2a^2} (1 - 5 \cos^2 I) \quad (3.3d)$$

$$\dot{\Omega} = \frac{3nC_{20}a_E^2}{2(1-e^2)^2a^2} \cos I \quad (3.3e)$$

$$\dot{M} = n - \frac{3nC_{20}a_E^2}{4(1-e^2)^{3/2}a^2} (3 \cos^2 I - 1) \quad (3.3f)$$

**Discussion** The flattening of the Earth has no effect on the shape and size of the orbit ( $a$  and  $e$ ). The inclination of the orbital plane remains constant, too ( $I$ ). There will be a precession of the orbital plane, though ( $\dot{\Omega}$ ). Within the orbit, the flattening effect is twofold: the perigee starts to precess ( $\dot{\omega}$ ) and the mean motion gets an additional term.

For a satellite at about 750 km height, following a near-circular orbit (e.g.  $e = 0.01$ ), the above equations become:

$$\dot{\omega} \approx 3.35 (5 \cos^2 I - 1) \text{ per day} \quad (3.4a)$$

$$\dot{\Omega} \approx -6.7 \cos I \text{ per day} \quad (3.4b)$$

$$\dot{M} \approx 14.4 + (3 \cos^2 I - 1) \text{ revolutions per day} \quad (3.4c)$$

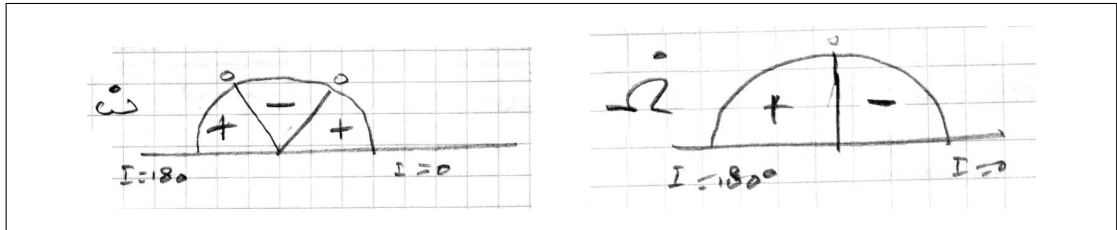


Figure 3.1

Some applications of the above formulae:

**Polar orbit** For a polar orbit ( $I = 90^\circ$ ), the equatorial bulge has no effect on the ascending node. Its precession remains zero and the orbital plane keeps its orientation in inertial space.

**Sun-synchronous orbit** For remote sensing purposes (lighting angle) and engineering purposes (no moving solar paddles, no Earth shadow transitions) a sun-synchronous orbit is very useful. Sun-synchronicity is attained if the orbital plane precession is equal to the Earth's rotation around the sun, i.e.  $\dot{\Omega} = 2\pi/\text{year}$ , which is nearly

1° per day. For the above numerical example, this is achieved at the near-polar retrograde inclination of 98°5.

**Critical inclination** Perigee precession does not occur if  $5 \cos^2 I = 1$ , which leads to  $I = 63^\circ 43'$ . This inclination is used in altimetry, for instance. An interesting use of this property is made by the Russian system of Molniya communication satellites, which have a very large eccentricity and semi-major axis. The perigee at  $270^\circ$  is fixed by a critical inclination. Thus these satellites swing around the Southern hemisphere rapidly, after which they will be visible over the Northern hemisphere (Russia) for a long time.

**Repeat orbit** A repeat orbit performs  $\beta$  revolutions in  $\alpha$  days. The integers  $\alpha$  and  $\beta$  are relative primes, i.e. they have no common divisor. The semi-major axis of such an orbit can be estimated by Kepler's third law already:

$$\begin{aligned} n^2 a^3 &= GM \quad \text{with } n = \frac{2\pi\beta}{\alpha} \text{ day}^{-1} \\ \implies a &= \sqrt[3]{\frac{GM}{n^2}} \quad \left( \text{oder } n = \sqrt{\frac{GM}{a^3}} \right) \\ \frac{\alpha}{\beta} &= \frac{T_{\text{Tag}}}{T_{\text{Umlauf}}} = \frac{2\pi/\omega_E}{2\pi/n} = \frac{n}{\omega_E} \\ \implies n &= \frac{\beta}{\alpha} \omega_E = \frac{2\pi\beta}{\alpha} \text{ per day} \\ \implies \text{Einsetzen in } a &= \sqrt[3]{\frac{GM}{n^2}} \end{aligned}$$

For a more precise estimate, the ratio between  $\dot{u} = \dot{\omega} + \dot{\nu}$  and  $\omega_E - \dot{\Omega}$  must be considered, in which  $u$  is the argument of latitude and  $\omega_E$  the rotation rate of the Earth.

$$\frac{\alpha}{\beta} = \frac{\dot{\omega} + \dot{M}}{\omega_E - \dot{\Omega}} = \frac{T_{\text{Knotentag}}}{T_{\text{Umlauf}}} \text{ etc.}$$

## 3.2 Canonical Equations

### 3.2.1 Newton equations

-  $\nabla R_{2,0}$

$$\frac{\partial R_{2,0}(x, y, z)}{\partial x}$$

- Newton - kanonische

$$\ddot{\mathbf{r}} = \nabla V \quad 3 \times 2. \text{ Ordnung GDGL}$$

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \nabla V \quad 6 \times 1. \text{ Ordnung GDGL}$$

$$F = T - V = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} - V(\mathbf{r}) \quad \begin{array}{l} = \text{force function} \\ = \text{Gesamtenergie} \\ = \text{Hamiltonian} \end{array}$$

$$\dot{\mathbf{r}} = \frac{\partial F}{\partial \mathbf{v}} = \frac{\partial T}{\partial \mathbf{v}}, \quad \dot{\mathbf{v}} = -\frac{\partial F}{\partial \mathbf{r}}$$

oder

$$\begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial F / \partial r_1 \\ \partial F / \partial r_2 \\ \partial F / \partial r_3 \\ \partial F / \partial v_1 \\ \partial F / \partial v_2 \\ \partial F / \partial v_3 \end{pmatrix}$$

Kanonische Gleichungen,  $\mathbf{r}, \mathbf{v}$  kanonische Variablen

$$\dot{q}_i = \frac{\partial M}{\partial p_i}, \quad i = 1, 2, 3 \quad \text{generalized coordinates}$$

$$\dot{p}_i = -\frac{\partial M}{\partial q_i} \quad \text{generalized moments}$$

### 3.2.2 Delaunay

### 3.2.3 Hill

## 3.3 Gauss form

Frage: Was wenn Kraft nicht  $\nabla V$  ist?

- nicht-grav. Kräfte (Solardruck, atm. Reibung)

- LPE funktioniert nicht mehr



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**⇒ Gauss-Form der LPE**

$$\text{Force } \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \begin{array}{l} \text{- quasi along-track - komplementaer (RMS)} \\ \text{- cross-track} \quad \quad \quad \text{- } \mathbf{L} \\ \text{- radial} \quad \quad \quad \quad \quad \text{- } \mathbf{r} \end{array}$$

$$\dot{a} = \frac{2}{n\sqrt{1-e^2}} \left( e \sin \nu f_3 + \frac{p}{r} f_1 \right)$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{na} (\sin \nu f_3 + (\cos E + \cos \nu) f_1)$$

$$\dot{I} = \frac{r}{nab} \cos(\omega + \nu) f_2$$

$$\dot{\Omega} = \frac{r}{nab \sin I} \sin(\omega + \nu) f_2$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{nae} \left( -\cos \nu f_3 + \left( \frac{r}{p} + 1 \right) \sin \nu f_1 \right) - \cos I \dot{\Omega}$$

$$\dot{M} = n - \frac{1}{na} \left( \frac{2r}{a} - \frac{1-e^2}{e} \cos \nu \right) f_3 - \frac{1-e^2}{nae} \left( 1 + \frac{r}{p} \right) \sin \nu f_1$$

with:

$$r = \frac{p}{1 + e \cos \nu} \implies \frac{p}{r} = 1 + e \cos \nu$$

$$p = a(1 - e^2) \text{ and } p = \frac{L^2}{GM}$$

$$\implies L = \sqrt{pGM} = \sqrt{a(1 - e^2)n^2 a^3} = nab$$

Diskussion:

- Wie aendert man  $a$ , oder  $I, \Omega$  etc.?
- Wie wirkt sich
- drag aus
- GOCE solar pressure
- Albedo

**Gauss-LPE**Approximation  $e \approx 0$ 

$$\dot{a} = \frac{2}{n} f_1$$

$$\dot{e} = \frac{1}{na} (\sin \nu f_3 + 2 \cos \nu f_1)$$

### 3 Equations of perturbed motion

$$\dot{I} = \frac{1}{na} \cos u f_2$$

$$\dot{\Omega} = \frac{1}{na \sin I} \sin u f_2$$

# NB: drag //  $\nu \neq$  along-track

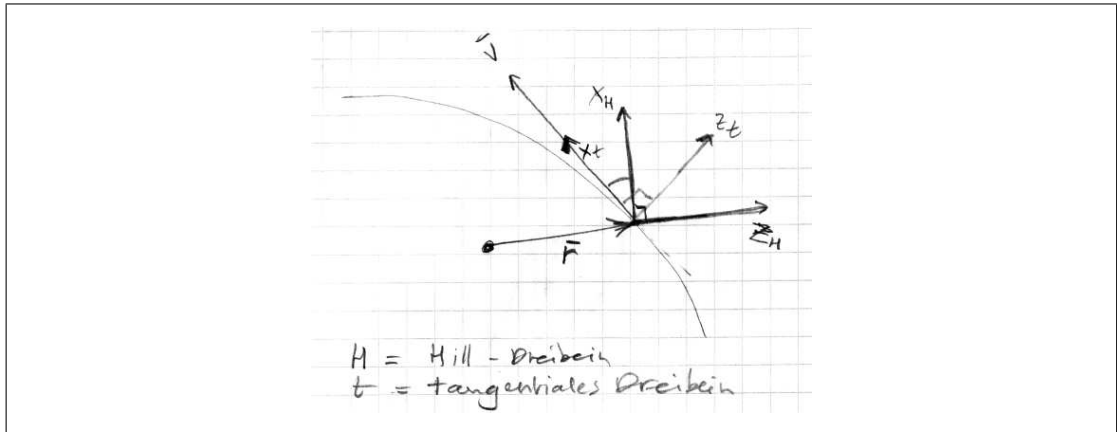


Figure 3.2

$$\mathbf{f}_t = R_2(??) \mathbf{f}_H \quad \tan \kappa = \frac{e \sin \nu}{1 + e \sin \nu}$$

$$= \frac{1}{\sqrt{1 + e^2 + 2e \cos \nu}} \begin{pmatrix} 1 + e \cos \nu & 0 & -e \sin \nu \\ 0 & \sqrt{1 + e^2 + 2e \cos \nu} & 0 \\ e \sin \nu & 0 & 1 + e \cos \nu \end{pmatrix}$$

Check matrix:

- $R(1,1) = R(3,3)$
- $R(1,3) = -R(3,1)$
- nullen und eins richtig
- Spalte ? =  $\sqrt{\frac{1+e^2 \cos^2 \nu + 2e \cos \nu + e^2 \sin^2 \nu}{1+e^2+2e \cos \nu}} = 1$
- etc.

Drag:

$$\mathbf{f}_t^{\text{drag}} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix}_t \implies \mathbf{f}_H^{\text{drag}} = R_2(-??) \mathbf{f}_t^{\text{drag}}$$

**LPE in Gauss form in tangential frame**

t-frame: 1: along-track, 2: cross-track, 3: quasi-radial

$$\begin{aligned}\dot{a} &= \frac{2a^2 v}{GM} f_1 \\ \dot{e} &= \frac{1}{v} \left( \frac{r}{a} \sin \nu f_3 + 2(e + \cos \nu) f_1 \right) \\ \dot{I} &= \frac{r}{L} \cos u f_2 \\ \dot{\Omega} &= \frac{r}{L \sin I} \sin u f_2 \\ \dot{\omega} &= \frac{1}{ev} \left( - \left( 2e + \frac{r}{a} \right) \cos \nu f_3 + 2 \sin \nu f_1 \right) - \frac{r \cos I}{L \sin I} \sin u f_2 \\ \dot{M} &= n + \frac{b}{a} \frac{1}{ev} \left( \frac{r}{a} \cos \nu f_3 - 2 \left( 1 + e^2 \frac{r}{p} \right) \sin \nu f_1 \right)\end{aligned}$$

with:

$$\begin{aligned}L &= na^2 \sqrt{1 - e^2} = nab \\ \frac{p}{r} &= 1 + e \cos \nu \\ v &= \frac{L}{p} \sqrt{1 + e^2 + 2e \cos \nu} \\ u &= \omega + \nu\end{aligned}$$

**Beispiel Hohmann Transfer**

$$\begin{aligned}v_1 &= 7560 \frac{\text{m}}{\text{s}} \\ v_{2(\text{peri})} &= 9902 \frac{\text{m}}{\text{s}}\end{aligned} \left. \vphantom{\begin{aligned}v_1 \\ v_{2(\text{peri})}\end{aligned}} \right\} \Delta v_{\text{perigee}}$$

$$\begin{aligned}v_{2(\text{apo})} &= 1639 \frac{\text{m}}{\text{s}} \\ v_3 &= 3075 \frac{\text{m}}{\text{s}}\end{aligned} \left. \vphantom{\begin{aligned}v_{2(\text{apo})} \\ v_3\end{aligned}} \right\} \Delta v_{\text{apogee}}$$

After 5 perigee boosts and 3 apogee boosts: Circular orbit at  $a = a_{\text{GEO}} - 5000\text{km}$ !

$$\begin{aligned}E_{\text{GEO}} &= -4.73 \frac{\text{km}^2}{\text{s}^2} \\ E_{\text{Artemis}} &= -5.36 \frac{\text{km}^2}{\text{s}^2}\end{aligned} \left. \vphantom{\begin{aligned}E_{\text{GEO}} \\ E_{\text{Artemis}}\end{aligned}} \right\} \Delta E = 0.63 \frac{\text{km}^2}{\text{s}^2}$$

Annahme:  $10 \frac{\mu\text{m}}{\text{s}^2}$  ion thrusters

$$\begin{aligned}\Delta E &= \int_s \mathbf{f} \cdot d\mathbf{s} = \int \underbrace{f_1}_{\text{along-track}} ds = \int f \underbrace{v}_{\sim 3100 \frac{\text{m}}{\text{s}}} dt \\ &\approx fv\Delta t \\ \implies \Delta t &= ?? \\ \dot{a} &= \dots \frac{\text{m}}{\text{day}} \\ \text{LPE Gauss } \dot{a} &= \frac{2}{n} f_1 = \dots \frac{\text{m}}{\text{day}}?\end{aligned}$$

## 4 A viable alternative: Hill Equations

The standard procedure in dynamic satellite geodesy is to develop a linear perturbation theory in terms of Kepler<sup>1</sup> elements. To this end, the Newton<sup>2</sup> equations of motion  $\ddot{\mathbf{r}} = \mathbf{a}$  are transformed into equations of motion of the form:

$$\dot{\mathbf{k}} = \mathbf{f}(\mathbf{k}), \quad \mathbf{k} = (a, e, I, \Omega, \omega, M).$$

These are the so-called Lagrange<sup>3</sup> planetary equations, a set of 6 first-order coupled non-linear ordinary differential equations (ODE). They are solved by noticing that the major gravitational perturbation is due to the dynamic flattening of the Earth (expressed by  $J_2$ ), causing the Kepler orbit to precess with  $\dot{\Omega}$ ,  $\dot{\omega}$  and  $\dot{M}$  all proportional to  $J_2$ . The non-linear ODE are linearized on this precessing or secular reference orbit. This is the procedure followed in (Kaula, 1966) and most other textbooks.

Here we will follow a different approach. Most satellites of geodetic interest are following a near-circular orbit. Therefore, we will use a set of equations that describes motion in a reference frame, that co-rotates with the satellite on a circular path. These are the Hill<sup>4</sup> equations, that were revived for geodetic purposes by O.L. Colombo, E.J.O. Schrama and others.

### 4.1 Acceleration in a rotating reference frame

Let us consider the situation of motion in a *rotating* reference frame and let us associate this rotating frame with a triad that is rotating uniformly on a nominal circular orbit, for the time being. Inertial coordinates, velocities and accelerations will be denoted with the index  $i$ . Satellite-frame quantities get the index  $s$ . Now suppose that a *time-dependent*

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<sup>1</sup>Johannes Kepler (1571–1630). Gave the first mathematical description of (planetary) orbits: *i*) Planets move on an elliptical orbit around the sun in one of the focal points, *ii*) The line between sun and planet sweeps equal areas in equal times, and *iii*) The ratio between the cube of the semi-major axis and the square of the revolution period is constant.

<sup>2</sup>Sir Isaac Newton (1642–1727).

<sup>3</sup>Comte Louis de Lagrange (1736–1813). French-Italian mathematician and astronomer.

<sup>4</sup>George William Hill (1838–1914), American mathematician. He developed his eponymous equations to describe lunar motion in his *Researches in the Lunar Theory* (1878), American Journal of Mathematics, vol. 1, pp. 5–26, 129–147, 245–260

rotation matrix  $R = R(\alpha(t))$ , applied to the inertial vector  $\mathbf{r}_i$ , results in the Earth-fixed vector  $\mathbf{r}_s$ . We would be interested in velocities and accelerations in the rotating frame. The time derivations must be performed in the inertial frame, though.

From  $R\mathbf{r}_i = \mathbf{r}_s$  we get:

$$\mathbf{r}_i = R^\top \mathbf{r}_s \quad (4.1a)$$

↓ time derivative

$$\dot{\mathbf{r}}_i = R^\top \dot{\mathbf{r}}_s + \dot{R}^\top \mathbf{r}_s \quad (4.1b)$$

↓ multiply by  $R$

$$\begin{aligned} R\dot{\mathbf{r}}_i &= \dot{\mathbf{r}}_s + R\dot{R}^\top \mathbf{r}_s \\ &= \dot{\mathbf{r}}_s + \Omega \mathbf{r}_s \end{aligned} \quad (4.1c)$$

The matrix  $\Omega = R\dot{R}^\top$  is called *Cartan*<sup>5</sup> *matrix*. It describes the rotation rate, as can be seen from the following simple 2D example with  $\alpha(t) = \omega t$ :

$$\begin{aligned} R &= \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \\ \Rightarrow \Omega &= \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \omega \begin{pmatrix} -\sin \omega t & -\cos \omega t \\ \cos \omega t & -\sin \omega t \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \end{aligned}$$

It is useful to introduce  $\Omega$ . In the next time differentiation step we can now distinguish between time dependent rotation matrices and time variable rotation rate. Let's pick up the previous derivation again:

↓ multiply by  $R^\top$

$$\dot{\mathbf{r}}_i = R^\top \dot{\mathbf{r}}_s + R^\top \Omega \mathbf{r}_s \quad (4.1d)$$

↓ time derivative

$$\begin{aligned} \ddot{\mathbf{r}}_i &= R^\top \ddot{\mathbf{r}}_s + \dot{R}^\top \dot{\mathbf{r}}_s + \dot{R}^\top \Omega \mathbf{r}_s + R^\top \dot{\Omega} \mathbf{r}_s + R^\top \Omega \dot{\mathbf{r}}_s \\ &= R^\top \ddot{\mathbf{r}}_s + 2\dot{R}^\top \dot{\mathbf{r}}_s + \dot{R}^\top \Omega \mathbf{r}_s + R^\top \dot{\Omega} \mathbf{r}_s \end{aligned} \quad (4.1e)$$

↓ multiply by  $R$

$$R\ddot{\mathbf{r}}_i = \ddot{\mathbf{r}}_s + 2\Omega \dot{\mathbf{r}}_s + \Omega \Omega \mathbf{r}_s + \dot{\Omega} \mathbf{r}_s \quad (4.1f)$$

This equation tells us that acceleration in the rotating  $e$ -frame equals acceleration in the inertial  $i$ -frame—in the proper orientation, though—when 3 more terms are added. The additional terms are called *inertial* accelerations. Analyzing (4.1f) we can distinguish the four terms at the right hand side:

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<sup>5</sup>Élie Joseph Cartan (1869–1951), French mathematician.

- $R\ddot{\mathbf{r}}_i$  is the inertial acceleration vector, expressed in the orientation of the rotating frame.
- $2\Omega\dot{\mathbf{r}}_s$  is the so-called *Coriolis* acceleration, which is due to motion in the rotating frame.
- $\Omega\Omega\mathbf{r}_s$  is the *centrifugal* acceleration, determined by the position in the rotating frame.
- $\dot{\Omega}\mathbf{r}_s$  is sometimes referred to as *Euler* acceleration or inertial acceleration of rotation. It is due to a non-constant rotation rate.

**Remark 4.1** Equation (4.1f) can be generalized to moving frames with time-variable origin. If the linear acceleration of the  $e$ -frame's origin is expressed in the  $i$ -frame with  $\ddot{\mathbf{b}}_i$ , the only change to be made to (4.1f) is  $R\ddot{\mathbf{r}}_i \rightarrow R(\ddot{\mathbf{r}}_i - \ddot{\mathbf{b}}_i)$ .

**Properties of the Cartan matrix  $\Omega$ .** Cartan matrices are skew-symmetric, i.e.  $\Omega^\top = -\Omega$ . This can be seen in the simple 2D example above already. But it also follows from the orthogonality of rotation matrices:

$$RR^\top = I \implies \frac{d}{dt}(RR^\top) = \underbrace{\dot{R}R^\top}_{\Omega^\top} + \underbrace{R\dot{R}^\top}_{\Omega} = 0 \implies \Omega^\top = -\Omega. \quad (4.2)$$

A second interesting property is the fact that multiplication of a vector with the Cartan matrix equals the cross product of the vector with a corresponding rotation vector:

$$\Omega\mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \quad (4.3)$$

This property becomes clear from writing out the 3 Cartan matrices, corresponding to the three independent rotation matrices:

$$\left. \begin{aligned} R_1(\omega_1 t) &\Rightarrow \Omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_1 \\ 0 & \omega_1 & 0 \end{pmatrix} \\ R_2(\omega_2 t) &\Rightarrow \Omega_2 = \begin{pmatrix} 0 & 0 & \omega_2 \\ 0 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{pmatrix} \\ R_3(\omega_3 t) &\Rightarrow \Omega_3 = \begin{pmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \right\} \xrightarrow{\text{general}} \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \quad (4.4)$$

Indeed, when a general rotation vector  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^\top$  is defined, we see that:

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The skew-symmetry (4.2) of  $\Omega$  is related to the fact  $\boldsymbol{\omega} \times \mathbf{r} = -\mathbf{r} \times \boldsymbol{\omega}$ .

**Exercise 4.1** Convince yourself that the above Cartan matrices  $\Omega_i$  are correct, by doing the derivation yourself.

Using property (4.3), the velocity (4.1c) and acceleration (4.1f) may be recast into the perhaps more familiar form:

$$R\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_s + \boldsymbol{\omega} \times \mathbf{r}_s \quad (4.5a)$$

$$R\ddot{\mathbf{r}}_i = \ddot{\mathbf{r}}_s + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}_s + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_s) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_s \quad (4.5b)$$

## 4.2 Hill equations

**Rotation.** As inertial system we will use the so-called *perifocal* system, which has its  $x_i y_i$ -plane in the orbital plane with the  $x_i$ -axis pointing towards the perigee. Thus the  $z_i$  axis is aligned with the angular momentum vector. This may not be the conventional inertial system, but it is a convenient one for the following discussion. If you don't like the perifocal frame you have to perform the following rotations first:

$$\mathbf{r}_i = R_3(\omega)R_1(I)R_3(\Omega)\mathbf{r}_{i_0},$$

with  $\Omega$  the right ascension of the ascending node (not to be mistaken for the Cartan matrix),  $I$  the inclination,  $\omega$  the argument of perigee (not to be mistaken for the rotation rate), and the index  $i_0$  referring to the conventional inertial system.

The  $s$ -frame will be rotating around the  $z_i = z_s$ -axis at a constant rotation rate  $n$  that we will later identify with a satellite's mean motion. Thus, the rotation angle is  $nt$ :

$$\mathbf{r}_s = R_3(nt)\mathbf{r}_i. \quad (4.6)$$

$$\Omega = \begin{pmatrix} 0 & -n & 0 \\ n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \dot{\Omega} = 0.$$



The three inertial accelerations, due to the rotation of the Earth, become:

$$\text{Coriolis:} \quad -2\Omega\dot{\mathbf{r}}_s = 2n \begin{pmatrix} \dot{y}_s \\ -\dot{x}_s \\ 0 \end{pmatrix} \quad (4.7a)$$

$$\text{centrifugal:} \quad -\Omega\Omega\mathbf{r}_s = n^2 \begin{pmatrix} x_s \\ y_s \\ 0 \end{pmatrix} \quad (4.7b)$$

$$\text{Euler:} \quad -\dot{\Omega}\mathbf{r}_s = \mathbf{0} \quad (4.7c)$$

**Translation.** Now let's introduce a *nominal orbit* of constant radius  $R$ , which should not be mistaken for the rotation matrix  $R$ . A satellite on this orbit would move with uniform angular velocity  $n$ , according to Kepler's third law:  $n^2 R^3 = GM$ .

The origin of the  $s$ -frame is now translated to the nominal orbit over the  $x_s$ -axis. While the frame is revolving on the nominal orbit, the  $x_s$  axis continuously points in the radial direction, the  $y_s$ -axis is in along-track, whereas the  $z_s$  axis points cross-track. As mentioned before, the translation induces an additional origin acceleration. Since we are dealing with circular motion this is a *centripetal* acceleration. In the orientation of the  $s$ -frame it is purely radial (in negative direction:)  $R\ddot{\mathbf{b}}_i = (-n^2 R, 0, 0)^\top$ .

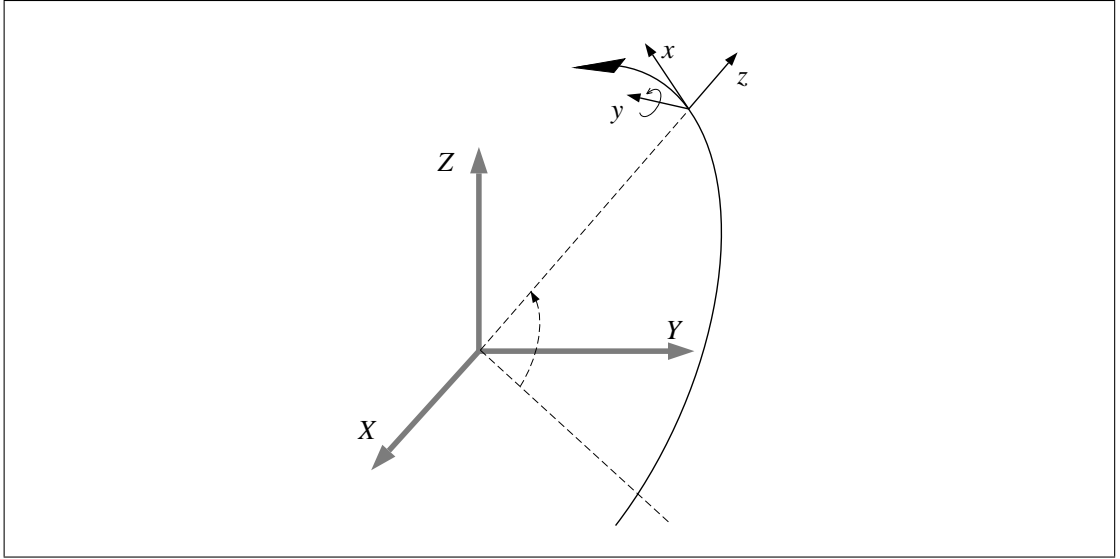
**Permutation.** In local frames, we usually want the  $z$ -coordinate in the vertical direction. Thus we now permute the coordinates according to fig. 4.1. At the same time we will drop the index  $s$ .

$$\begin{aligned} y_s &\rightarrow x = \text{along-track} \\ z_s &\rightarrow y = \text{cross-track} \\ x_s &\rightarrow z = \text{radial} \end{aligned}$$

**Hill equations: kinematics.** Notice that sofar we have only dealt with kinematics, i.e. a description of position, velocity and acceleration under the transformation from the inertial to the satellite frame. We do not have *equations of motion* yet, that will only come up as soon as we introduce dynamics (a force) as well.

Combining all the kinematic information we have, we arrive at the following:

$$\left. \begin{aligned} \ddot{x} + 2n\dot{z} - n^2x &= \\ \ddot{y} &= \\ \ddot{z} - 2n\dot{x} - n^2z - n^2R &= \end{aligned} \right\} R\ddot{\mathbf{r}}_i \quad (4.8)$$



**Figure 4.1:** The local orbital triad:  $x$  along-track,  $y$  cross-track and  $z$  radial

**Hill equations: dynamics.** Now let's turn to the right hand side. In inertial space, the equations of motion are simply Newton's equations. We assume that the force is composed of a central field term  $U(\mathbf{r}) = GM/r$ , with  $r = |\mathbf{r}|$  and a term that contains all other forces, both gravitational and non-gravitational:

$$\ddot{\mathbf{r}}_i = \nabla_i U(\mathbf{r}_i) + \mathbf{f}_i = -\frac{GM}{r^3} \mathbf{r}_i + \mathbf{f}_i. \quad (4.9)$$

According to (4.8) we need to rotate this with the matrix  $R$ . At the same time we will *linearize* this on the circular nominal orbit:

$$\begin{aligned} R\ddot{\mathbf{r}}_i &= \nabla_s U(\mathbf{r})|_{r=R} + \nabla_s^2 U(\mathbf{r})|_{r=R} \cdot \mathbf{r}_s + \mathbf{f}_s + \mathcal{O}(r_s^2) \\ &= -\frac{GM}{R^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{GM}{R^3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\ &= -n^2 \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix} + n^2 \begin{pmatrix} -x \\ -y \\ 2z \end{pmatrix} + \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}. \end{aligned}$$

In the last step we used Kepler's third again, i.e.  $n^2 R^3 = GM$ . Now inserting the linearized dynamics into (4.8) results in:

$$\begin{array}{rcl} \ddot{x} + 2n\dot{z} & = & f_x \\ \ddot{y} & + n^2 y & = f_y \\ \ddot{z} - 2n\dot{x} - 3n^2 z & = & f_z \end{array} \quad (4.10)$$

These are known as *Hill equations*. They describe satellite motion in a satellite frame that is co-rotating on a circular path at uniform speed  $n$ . Note that they are *approximated* equations of motion due to

- the linearization of  $\nabla U$  on the circular orbit,
- constant radius approximation of the gravity gradient tensor  $\nabla^2 U$ .

From (4.10) it is obvious that the motion in the orbital plane  $(x, z)$  is coupled. The cross-track motion equation is that of a harmonic oscillator.

### 4.3 Solutions of the Hill equations

The key advantage of Hill equations (HE) is that they are linear ordinary differential equations with constant coefficients. That means that we will be able to find an analytical solution. Thus we can find an exact solution to approximated equations of motion. This is in contrast to the Lagrange Planetary Equations. They are exact equations of motions that need to be solved by linear approximation.

The HE are second order ODEs. For this type of equations the following strategy solution usually works:

- i) Write the 3 second order equations as 6 first order equations. Actually, since the  $y$ -equation is decoupled one can setup two separate sets of first order equations.
- ii) Write the set of equations as  $\dot{\mathbf{u}} = A\mathbf{u}$ .
- iii) Perform eigenvalue decomposition on  $A = Q\Lambda Q^{-1}$ .
- iv) Transform  $\dot{\mathbf{u}} = A\mathbf{u}$  into  $Q^{-1}\dot{\mathbf{u}} = \Lambda Q^{-1}\mathbf{u}$  or simply  $\dot{\mathbf{v}} = \Lambda\mathbf{v}$ .
- v) Solve the decoupled equations by:  $v_n = c_n e^{\lambda_n t}$  or in vector-matrix form:  $\mathbf{v} = e^{\Lambda t} \mathbf{c}$ .
- vi) Transform back using the eigenvector matrix  $Q$ :  $\mathbf{u} = Q\mathbf{v} = Qe^{\Lambda t} \mathbf{c}$  or in index form  $\mathbf{u} = \sum_n c_n e^{\lambda_n t} \mathbf{q}_n$ .

The problem is: that doesn't work for the coupled in-plane equations. One can set up

the  $4 \times 4$  matrix  $A$ :

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2n \\ 0 & 3n^2 & 2n & 0 \end{pmatrix}$$

The characteristic polynomial of this matrix is:

$$\det(A - \lambda I) = \lambda^2(\lambda^2 + n^2) = 0.$$

This gives the 4 eigenvalues  $\lambda_1 = in$ ,  $\lambda_2 = -in$  and  $\lambda_{3,4} = 0$ . This is where the trouble starts. For the first two eigenvalues one can find eigenvectors. To the double zero-eigenvalue, however, one can only find a single eigenvector. In mathematical terms: the algebraic multiplicity (2 eigenvalues) is larger than the geometric multiplicity (1 eigenvalue).

For this type of pathological matrices there is a way out: the *Jordan*<sup>6</sup> decomposition. Where the eigenvalue decomposition achieves a full decoupling or diagonalization, the Jordan decomposition tries to decouple as much as possible. That usually involves putting the number 1 in the first (few) off-diagonals. In terms of solutions one can expect to see terms with  $te^{\lambda nt}$ ,  $t^2e^{\lambda nt}$ , etc. next to the usual  $e^{\lambda nt}$ . In our case, with  $\lambda_{3,4} = 0$ , we can therefore expect to see terms that are linear in time.

### 4.3.1 The homogeneous solution

We start with the homogeneous Hill equations, i.e. the non-perturbed equations:

$$\begin{aligned} \ddot{x} + 2n\dot{z} &= 0 \\ \ddot{y} + n^2y &= 0 \\ \ddot{z} - 2n\dot{x} - 3n^2z &= 0 \end{aligned} \quad (4.11)$$

Their solution reads:

$$x(t) = \frac{2}{n}\dot{z}_0 \cos nt + \left(\frac{4}{n}\dot{x}_0 + 6z_0\right) \sin nt - (3\dot{x}_0 + 6nz_0)t + x_0 - \frac{2}{n}\dot{z}_0 \quad (4.12a)$$

$$y(t) = y_0 \cos nt + \frac{\dot{y}_0}{n} \sin nt \quad (4.12b)$$

$$z(t) = \left(-\frac{2}{n}\dot{x}_0 - 3z_0\right) \cos nt + \frac{\dot{z}_0}{n} \sin nt + \frac{2}{n}\dot{x}_0 + 4z_0 \quad (4.12c)$$

The solution mainly consists of periodic motion at the orbit frequency  $n$ . But indeed the  $x$ -component contains a term linear in  $t$ . Again, it is seen that  $x$ - and  $z$ -motion is coupled, whereas  $y$  behaves as a pure oscillator independent from the other terms.

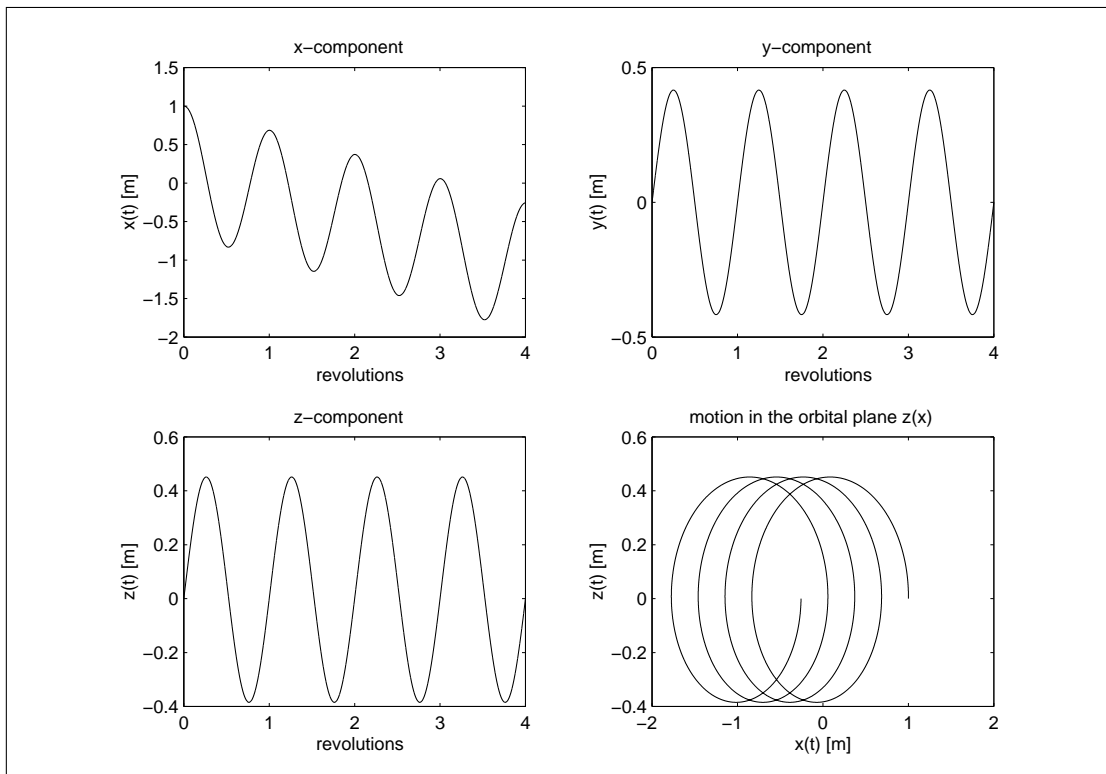
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<sup>6</sup>Marie Ennemond Camille Jordan (1838–1922), French mathematician.

The amplitudes of the sines and cosines as well as the constant terms and the drift are purely dependent on the initial state elements. Note that these initial state elements are given in the co-rotating satellite frame. Thus, the homogeneous solution (4.12a) can be used for initial state problems, e.g.:

- docking manoeuvres,
- $\Delta v$  thrusts,
- configuration flight design.

The homogeneous response is visualized in fig. 4.2.



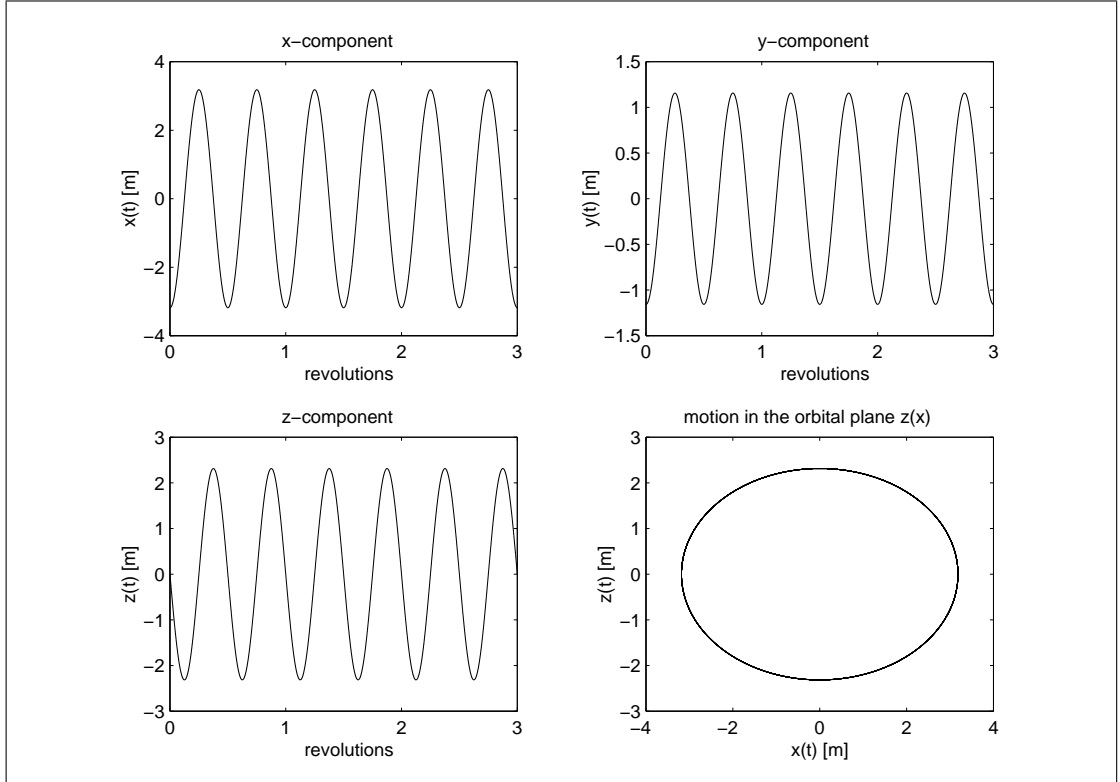
**Figure 4.2:** Homogeneous solution.

### 4.3.2 The particular solution

Suppose the orbit is perturbed by a force that can be decomposed into a Fourier series. This is for instance the case with gravitational forces, cf. next chapter. Since the HE is a system of linear ODE's an forcing (the input) at a certain frequency will result in an orbit perturbation (the output) at the same frequency. Thus we only need to investigate

the behaviour of the equations of motion at one specific disturbing frequency  $\omega$  (not to be mistaken for rotation rate nor for argument of perigee). Then we can apply the superposition principle, or spectral synthesis, to achieve a full solution.

$$\begin{aligned} \ddot{x} + 2n\dot{z} &= A_x \cos \omega t + B_x \sin \omega t \\ \ddot{y} + n^2 y &= A_y \cos \omega t + B_y \sin \omega t \\ \ddot{z} - 2n\dot{x} - 3n^2 z &= A_z \cos \omega t + B_z \sin \omega t \end{aligned} \quad (4.13)$$



**Figure 4.3:** Particular solution with disturbance at  $\omega = 2n$ .

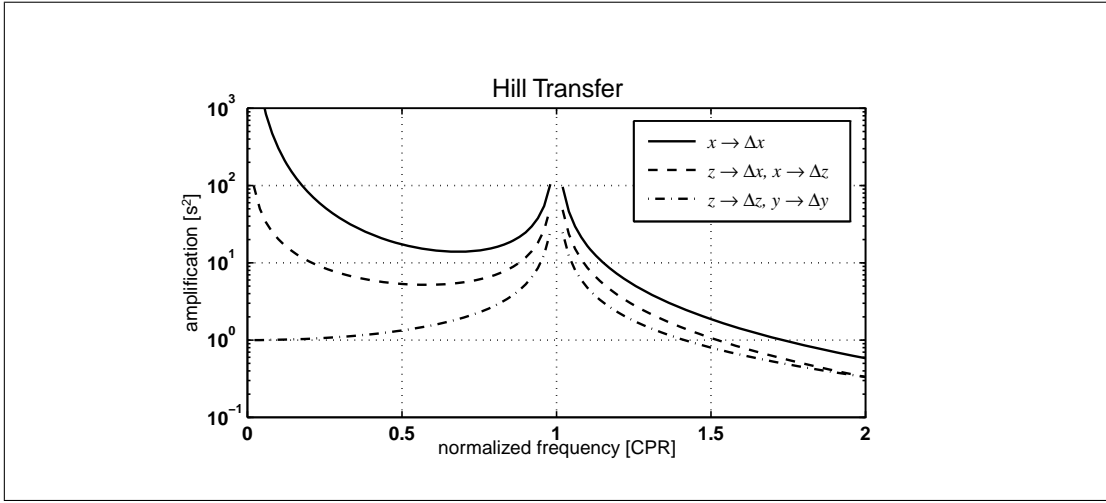
The solution to (4.13) reads:

$$x(t) = \frac{(3n^2 + \omega^2)A_x + 2\omega n B_z}{\omega^2(n^2 - \omega^2)} \cos \omega t + \frac{(3n^2 + \omega^2)B_x - 2\omega n A_z}{\omega^2(n^2 - \omega^2)} \sin \omega t \quad (4.14a)$$

$$y(t) = \frac{A_y}{n^2 - \omega^2} \cos \omega t + \frac{B_y}{n^2 - \omega^2} \sin \omega t \quad (4.14b)$$

$$z(t) = \frac{\omega A_z - 2n B_x}{\omega(n^2 - \omega^2)} \cos \omega t + \frac{\omega B_z + 2n A_x}{\omega(n^2 - \omega^2)} \sin \omega t \quad (4.14c)$$

Again one can see that the two components in the orbital plane,  $x$  and  $z$  are coupled. Inspecting the denominators in the perturbation amplitudes, we notice a huge amplification close to the frequencies  $\omega = 0$  and  $\omega = \pm n$ . This amplification is called *resonance*. The several resonances are visualized in fig. 4.4. It occurs if the satellite is excited at the zero frequency (DC) or at the orbital frequency itself. These are the eigenfrequencies of the system, that were already identified by the eigenvectors  $-in$ ,  $0$ , and  $in$ . For these frequencies the solution becomes invalid and we will seek another solution for resonant forcing later on.



**Figure 4.4:** Resonances of the Hill equations.

### 4.3.3 The complete solution

A complete solution will consist of a combination of particular and homogeneous solution. The homogeneous solution lies in the null space of the set of ODE. Thus one can always add the homogeneous solution (4.12a) without changing the right hand side of equations (4.13). The complete solutions reads:

$$\begin{aligned}
 x(t) = & \left( 2 \frac{\dot{z}_0}{n} - \frac{4A_x}{n^2 - \omega^2} - \frac{2\omega B_x}{n(n^2 - \omega^2)} \right) \cos nt \\
 & + \left( 6z_0 + 4 \frac{\dot{x}_0}{n} + \frac{2A_z}{n^2 - \omega^2} - \frac{4\omega B_x}{n(n^2 - \omega^2)} \right) \sin nt \\
 & + \frac{1}{\omega^2(n^2 - \omega^2)} \left( (3n^2 + \omega^2)A_x + 2\omega n B_z \right) \cos \omega t
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\omega^2(n^2 - \omega^2)} \left( (3n^2 + \omega^2)B_x + 2\omega n A_z \right) \sin \omega t \\
 & + \left( -6nz_0 - 3\frac{B_x}{\omega} - 3\dot{x}_0 \right) t + x_0 - 3\frac{A_x}{\omega^2} - 2\frac{B_z}{\omega n} - 2\frac{\dot{z}_0}{n}
 \end{aligned} \tag{4.15a}$$

$$\begin{aligned}
 y(t) & = \left( y_0 - \frac{A_y}{n^2 - \omega^2} \right) \cos nt + \left( \frac{\dot{y}_0}{n} - \frac{\omega B_y}{n(n^2 - \omega^2)} \right) \sin nt \\
 & + \frac{1}{n^2 - \omega^2} (A_y \cos \omega t + B_y \sin \omega t)
 \end{aligned} \tag{4.15b}$$

$$\begin{aligned}
 z(t) & = \left( -3z_0 - 2\frac{\dot{x}_0}{n} - \frac{A_z}{n^2 - \omega^2} + \frac{2\omega B_x}{n(n^2 - \omega^2)} \right) \cos nt \\
 & + \left( \frac{\dot{z}_0}{n} - \frac{2A_x}{n^2 - \omega^2} - \frac{\omega B_z}{n(n^2 - \omega^2)} \right) \sin nt \\
 & + \frac{1}{\omega(n^2 - \omega^2)} (\omega A_z - 2nB_x) \cos \omega t \\
 & + \frac{1}{\omega(n^2 - \omega^2)} (\omega B_z + 2nA_x) \sin \omega t + 4z_0 + 2\frac{B_x}{n\omega} + 2\frac{\dot{x}_0}{n}
 \end{aligned} \tag{4.15c}$$

As can be seen in the solution (4.15a) and in fig. 4.5 the complete solution is a superposition of two components: one at the orbital frequency  $n$  and one at the disturbing frequency  $\omega$ .

#### 4.3.4 The resonant solution

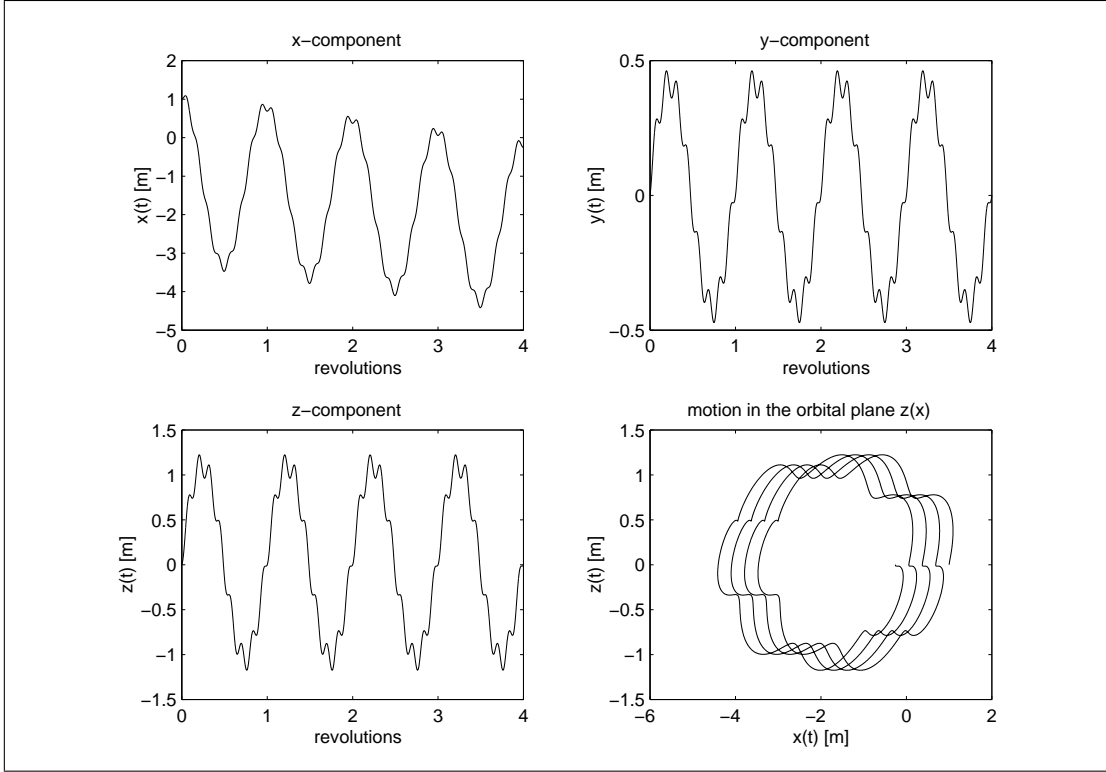
As mentioned before, the amplification for disturbances at  $\omega \rightarrow -n, 0, +n$  becomes infinite. This is only a mathematical shortcoming of our solution so far. In order to investigate resonance, we have to assume a forcing at the resonant frequencies. The corresponding Hill equations read:

$$\begin{aligned}
 \ddot{x} + 2n\dot{z} & = A_x \cos nt + B_x \sin nt + C_x \\
 \ddot{y} + n^2 y & = A_y \cos nt + B_y \sin nt + C_y \\
 \ddot{z} - 2n\dot{x} - 3n^2 z & = A_z \cos nt + B_z \sin nt + C_z
 \end{aligned} \tag{4.16}$$

These ODE are solved by:

$$\begin{aligned}
 x(t) & = \left( \frac{1}{n^2} (2n\dot{z}_0 - 4C_x + 3A_x + 2B_z) + \frac{1}{n} (-2B_x + A_z) t \right) \cos nt \\
 & + \left( \frac{1}{n^2} (6n^2 z_0 + 4n\dot{x}_0 + 5B_x + 2C_z - A_z) + \frac{1}{n} (2A_x + B_z) t \right) \sin nt
 \end{aligned}$$





**Figure 4.5:** Complete solution with arbitrary  $\omega$ .

$$\begin{aligned}
 & + \frac{1}{n^2}(n^2x_0 - 2n\dot{z}_0 + 4C_x - 3A_x - 2B_z) \\
 & + \frac{1}{n}(-6n^2z_0 - 3n\dot{x}_0 - B_x - 2C_z)t - \frac{3}{2}C_x t^2
 \end{aligned} \tag{4.17a}$$

$$y(t) = \left(y_0 - \frac{C_y}{n^2} - \frac{1}{2n}B_y t\right) \cos nt + \left(\frac{\dot{y}_0}{n} + \frac{B_y}{2n^2} + \frac{1}{2n}A_y t\right) \sin nt + \frac{1}{n^2}C_y \tag{4.17b}$$

$$\begin{aligned}
 z(t) = & \left(\frac{1}{n^2}(-3n^2z_0 - 2n\dot{x}_0 - 2B_x - C_z) - \frac{1}{2n}(2A_x + B_z)t\right) \cos nt \\
 & + \left(\frac{1}{2n^2}(2n\dot{z}_0 - 4C_x + 2A_x + B_z) + \frac{1}{2n}(-2B_x + A_z)t\right) \sin nt \\
 & + \frac{1}{n^2}(4n^2z_0 + 2n\dot{x}_0 + 2B_x + C_z) + \frac{2}{n}C_x t
 \end{aligned} \tag{4.17c}$$

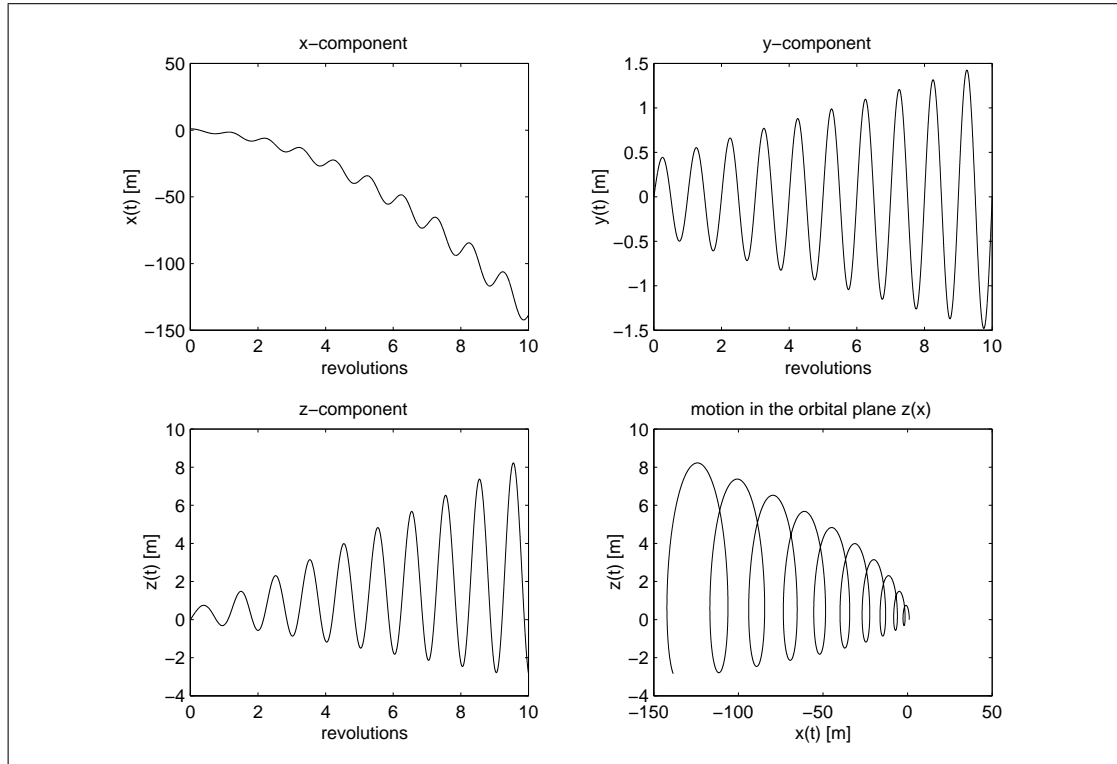
This solution contains amplitudes that grow linearly in time, see also fig. 4.6. This is characteristic for resonance. In terms of Kepler elements, these growing amplitudes

#### 4 A viable alternative: Hill Equations

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express secular orbit elements  $\dot{\omega}$ ,  $\dot{\Omega}$ ,  $\dot{M}$ .

The resonant equations are useful to investigate the behaviour of satellites under non-gravitational forces like air drag or solar radiation pressure.



**Figure 4.6:** Resonant solution.

## 5 The gravitational potential and its representation

The Lagrange Planetary equations (LPE) require the partial derivatives of the force function—or Hamiltonian—to all Kepler elements, i.e.  $\nabla_{\mathbf{s}}F$ . Thus, the main objective in this chapter is to transform  $V(r, \theta, \lambda)$  into  $V(a, e, I, \Omega, \omega, M) = V(\mathbf{s})$ .

The Gauss version of the LPE requires forces in a local satellite frame, either  $\mathbf{e}_s$  or  $\mathbf{e}_t$ . These can be generated by taking suitable derivatives of  $V(\mathbf{s})$ . Second derivatives in the local satellite frame will be presented, too. They are needed for gravity gradiometry.

### 5.1 Representation on the sphere

The gravitational potential is usually represented in a spherical harmonic. Such a representation turns out to be of advantage, since spherical harmonics possess the following properties:

- orthogonality,
- global support,
- harmonicity.

Because the geopotential fulfills the Laplace equation  $\Delta V = 0$  outside the masses, the harmonicity of the spherical harmonics makes them natural base functions to  $V$ . Their orthogonality allows the analysis of the coefficients of the base functions.

For reasons of compactness *complex-valued* quantities will be employed here:

$$V(r, \theta, \lambda) = \frac{GM}{R} \sum_{l=0}^{\infty} \left(\frac{R_E}{r}\right)^{l+1} \sum_{m=-l}^l K_{lm} Y_{lm}(\theta, \lambda), \quad (5.1)$$

in which

$r, \theta, \lambda$  = radius, co-latitude, longitude

$R$  = Earth's equatorial radius

$GM$  = gravitational constant times Earth's mass

$Y_{lm}(\theta, \lambda)$  = surface spherical harmonic of degree  $l$  and order  $m$

$K_{lm}$  = spherical harmonic coefficient, corresponding to  $Y_{lm}(\theta, \lambda)$ .

The coefficients  $K_{lm}$  constitute the *spherical harmonic spectrum* of the function  $V$ . They are the parameters of the gravitational field. The surface spherical harmonics  $Y_{lm}(\theta, \lambda)$  are defined in the following way:

$$Y_{lm}(\theta, \lambda) = P_{l,|m|}(\cos \theta) e^{im\lambda}. \quad (5.2)$$

It follows from this definition that for the complex conjugated it holds:  $Y_{lm}^* = Y_{l,-m}$ . Without explicitly using overbars, we assume that all complex quantities are (fully) normalized by the factor:

$$N_{lm} = \sqrt{(2l+1) \frac{(l-m)!}{(l+m)!}}. \quad (5.3)$$

Unnormalized spherical harmonic functions are multiplied by this factor to make them normalized. Unnormalized spherical harmonic coefficients are divided by (5.3). The orthogonality of the base functions is expressed by:

$$\frac{1}{4\pi} \iint_{\sigma} Y_{l_1 m_1}(\theta, \lambda) Y_{l_2 m_2}^*(\theta, \lambda) d\sigma = \delta_{l_1 l_2} \delta_{m_1 m_2}. \quad (5.4)$$

**Remark 5.1 (Normalization conventions)** *In literature, the factor  $\frac{1}{4\pi}$  is sometimes taken care of in the normalization factor by incorporating a term  $\sqrt{4\pi}$ . Another difference between normalization factors, found in literature, is a factor  $(-1)^m$ . It is often used implicitly in the definition of the Legendre functions.*

In geodesy, one usually employs real-valued base functions and coefficients, cf. (Heiskanen & Moritz, 1967). The series (5.1) would become:

$$V(r, \theta, \lambda) = \frac{GM}{R} \sum_{l=0}^{\infty} \left( \frac{R_E}{r} \right)^{l+1} \sum_{m=0}^l (C_{lm} \cos m\lambda + S_{lm} \sin m\lambda) P_{lm}(\cos \theta), \quad (5.5)$$

with normalization factor:

$$N_{lm} = \sqrt{(2 - \delta_{m0})(2l+1) \frac{(l-m)!}{(l+m)!}}. \quad (5.6)$$

The real- and complex-valued spherical harmonic coefficients, each with their own normalization, are linked by:

$$K_{lm} = \begin{cases} \frac{1}{2}(C_{lm} - iS_{lm}), & m > 0 \\ C_{lm}, & m = 0 \\ \frac{1}{2}(C_{lm} + iS_{lm}), & m < 0 \end{cases}, \quad (5.7)$$

such that  $K_{lm} = K_{l,-m}^*$ . Now it is easy to demonstrate the equality between complex and real-valued series expansions. If we ignore the dimensioning factor  $GM/R$ , the upward continuation term and the arguments of the spherical harmonics, we can write:

$$\begin{aligned}
 V &= \sum_l \sum_{m=-l}^l K_{lm} Y_{lm} \\
 &= \sum_l \sum_{m=0}^l K_{lm} Y_{lm} + K_{l,-m} Y_{l,-m} \\
 &= \sum_l \sum_{m=0}^l K_{lm} Y_{lm} + K_{lm}^* Y_{lm}^* \\
 &= \sum_l \sum_{m=0}^l K_{lm} Y_{lm} + (K_{lm} Y_{lm})^* \\
 &= \sum_l \sum_{m=0}^l 2\Re \{K_{lm} Y_{lm}\} \\
 &= \sum_l \sum_{m=0}^l 2 \frac{1}{2} \Re \{(C_{lm} - iS_{lm})(\cos m\lambda + i \sin m\lambda)\} P_{lm}(\cos \theta) \\
 &= \sum_l \sum_{m=0}^l (C_{lm} \cos m\lambda + S_{lm} \sin m\lambda) P_{lm}(\cos \theta)
 \end{aligned}$$

We made a minor mistake in the second line for the case  $m = 0$ , that could have been repaired explicitly by dividing by  $(1 + \delta_{m0})$ . However, the definition (5.7) already takes care of this. The opposite mistake is made in the second last line.

**Remark 5.2 (Complex vs. real)** *From the derivations above the benefits of a series expansion in complex quantities is obvious: compactness and transparency of formulas. An added benefit in the next section will be the transformation properties of spherical harmonics under rotation of the coordinate system. Such transformation properties would be extremely laborious in real notation.*

## 5.2 Representation in Kepler elements

In order to transform the potential into a function of Kepler elements, two steps are required:

- i) Rotate the spherical harmonics from the earth-fixed system into a coordinate system such that the orbit plane becomes the new equator and the new  $x$ -axis points towards the satellite. The following Euler rotation sequence is required:

$$R_{313}(\Omega - \text{GAST}, I, \omega + \nu) = R_3(\omega + \nu)R_1(I)R_3(\Omega - \text{GAST}) , \text{ or}$$

$$R_{313}(\Lambda, I, u) = R_3(u)R_1(I)R_3(\Lambda) ,$$

- ii) Express  $(R/r)^{l+1}e^{ik\nu}$  as a Fourier series in the mean anomaly  $M$ , multiplied by  $(R/a)^{l+1}$ .

**Step 1: Rotation of spherical harmonics.** If we rotate the coordinate system around the 3<sup>rd</sup> axis over an angle  $\alpha$ ,  $R_3(\alpha)$ , the coordinates themselves change as:

$$\theta' = \theta , \text{ and } \lambda' = \lambda - \alpha .$$

Under this rotation, surface spherical harmonics transform as:

$$Y_{lm}(\theta, \lambda) = P_{lm}(\cos \theta)e^{im\lambda} = P_{lm}(\cos \theta')e^{im(\lambda' + \alpha)} = Y_{lm}(\theta', \lambda')e^{im\alpha} . \quad (5.8)$$

Two of the three rotations can be dealt with now.

For rotations  $R_2$  and  $R_1$  things are not that simple. From representation theory we know that the transformation of a spherical harmonic  $Y_{lm}(\theta, \lambda)$  of a specific degree  $l$  and order  $m$  in one frame requires all spherical harmonics  $Y_{lk}(\theta', \lambda')$  of that same degree over all possible orders  $-l \leq k \leq l$  in the rotated frame in a certain linear way. The linear mapping is expressed by *representation coefficients*  $d_{lmk}$  that are a function of the rotation angle. For a rotation  $R_2(\alpha)$  we have the following transformation:

$$Y_{lm}(\theta, \lambda) = \sum_{k=-l}^l d_{lmk}(\alpha)Y_{lk}(\theta', \lambda') , \quad (5.9)$$

with

$$d_{lmk}(\alpha) = \left[ \frac{(l+k)!(l-k)!}{(l+m)!(l-m)!} \right]^{\frac{1}{2}} \sum_{t=t_1}^{t_2} \binom{l+m}{t} \binom{l-m}{l-k-t} (-1)^t c^{2l-a} s^a ,$$

in which  $c = \cos \frac{1}{2}\alpha$ ,  $s = \sin \frac{1}{2}\alpha$ ,  $a = k - m + 2t$ ,  $t_1 = \max(0, m - k)$  and  $t_2 = \min(l - k, l + m)$ .

Note that (5.8) can be cast into a similar form when we use Kronecker deltas:

$$Y_{lm}(\theta, \lambda) = \sum_{k=-l}^l \delta_{mk} e^{im\alpha} Y_{lk}(\theta', \lambda') .$$

Instead of a full  $(2l + 1) \times (2l + 1)$  linear system we have a diagonal matrix only.

Since we need to perform the rotation  $R_1(I)$ , (5.9) needs to be revised. A rotation around the 1<sup>st</sup> axis is achieved by a rotation around the 2<sup>nd</sup> axis if we properly pre- and postrotate by  $R_3(\pm\frac{1}{2}\pi)$ :

$$R_1(\alpha) = R_3(\frac{1}{2}\pi)R_2(\alpha)R_3(-\frac{1}{2}\pi).$$

Note that the rotation sequence is read from right to left. A spherical harmonic transforms under  $R_1(\alpha)$  therefore as follows:

$$Y_{lm}(\theta, \lambda) = \sum_{k=-l}^l e^{-im\frac{1}{2}\pi} d_{lmk}(\alpha) e^{ik\frac{1}{2}\pi} Y_{lk}(\theta', \lambda') = \sum_{k=-l}^l i^{k-m} d_{lmk}(\alpha) Y_{lk}(\theta', \lambda'). \quad (5.10)$$

In summary:

$$\begin{aligned} \mathbf{r}' &= R_3(u)R_1(I)R_3(\Lambda)\mathbf{r} \\ &= R_3(u + \frac{1}{2}\pi)R_2(I)R_3(\Lambda - \frac{1}{2}\pi)\mathbf{r} \end{aligned} \quad (5.11a)$$

$$\implies Y_{lm}(\theta, \lambda) = \sum_{k=-l}^l D_{lmk}(\Lambda, I, u) Y_{lk}(\theta', \lambda'), \quad (5.11b)$$

$$\text{with } D_{lmk}(\Lambda, I, u) = i^{k-m} d_{lmk}(I) e^{i(ku + m\Lambda)}. \quad (5.11c)$$

**New coordinates.** Using the time-variable elements  $u(t)$  and  $\Lambda(t)$ , the rotation sequence will keep the new  $x$ -axis pointing to the satellite. Its orbital plane will instantaneously coincide with a new equator. The satellite's coordinates reduce to  $\theta' = \frac{1}{2}\pi$  and  $\lambda' = 0$ , so that  $Y_{lk}(\theta', \lambda') = P_{lk}(0)$ . In principle the third rotation could have been omitted such that the representation coefficient  $D_{lmk}(\Lambda, I, 0)$  should have been used in (5.11c). In that case the longitude in the new frame would have been  $\lambda' = u$ , leading to the same expression. In both cases the satellite is always on the rotated equator. In the second interpretation the argument of latitude would become the new longitude. In this view the name *argument of latitude* his highly misplaced.

Inserting the transformation (5.11b) and the representation coefficients (5.11c) into (5.1), combined with  $\theta' = \frac{1}{2}\pi, \lambda' = 0$ :

$$V(r, u, \Lambda, I) = \frac{GM}{R} \sum_{l=0}^{\infty} \left(\frac{R_E}{r}\right)^{l+1} \sum_{m=-l}^l \sum_{k=-l}^l K_{lm} i^{k-m} d_{lmk}(I) P_{lk}(0) e^{i(ku + m\Lambda)}. \quad (5.12)$$

**Inclination functions.** As a simplification a so-called *inclination function* is introduced:

$$F_{lmk}(I) = i^{k-m} d_{lmk}(I) P_{lk}(0), \quad (5.13)$$

so that the along-orbit potential (5.12) is finally reduced to the series:

$$V(r, u, I, \Lambda) = \frac{GM}{R} \sum_{l=0}^{\infty} \left( \frac{R_E}{r} \right)^{l+1} \sum_{m=-l}^l \sum_{k=-l}^l K_{lm} F_{lmk}(I) e^{i(ku + m\Lambda)}. \quad (5.14)$$

The inclination functions (5.13) differ from Kaula's functions  $F_{lmp}(I)$  (Kaula, 1966) in the following aspects:

- they are complex,
- they are normalized by the factor (5.3) (though written here without overbar),
- they make use of the index  $k$ .

The 3<sup>rd</sup> index of Kaula's inclination function,  $p$ , is due to the following. The inclination function  $F_{lmk}(I)$  contains the equatorial Legendre function  $P_{lk}(0)$ . Legendre functions  $P_{lm}(x)$  are either even or odd functions on the domain  $x \in [-1; 1]$  for  $(l - m)$  even or odd, respectively. Thus, if  $(l - m)$  is odd,  $P_{lk}(0)$  will be zero and the whole inclination function becomes zero. Consequently the  $k$ -summation can be performed in steps of 2:  $\sum_{k=-l, 2}^l$ . This fact allows the introduction of another index:  $p = \frac{1}{2}(l - k)$  or  $k = l - 2p$ , which yields the summation  $\sum_{p=0}^l$ .

**Remark 5.3 ( $p$  vs.  $k$ )** *The  $p$ -index has two advantages: it is positive and it runs in unit steps. The third summation in (5.14) becomes  $\sum_{p=0}^l$ . The major disadvantage is that it does not have the meaning of spherical harmonic order (or azimuthal order) in the rotated system anymore. The index  $p$  is not a wavenumber, such as  $k$ . Thus, symmetries are lost, and formulae become more complicated. For instance  $\exp(i(ku + m\Lambda))$  must be written as  $\exp(i((l - 2p)u + m\Lambda))$ . The angular argument seems to depend on 3 indices in that case.*

**Step 2: Eccentricity functions.** So far, we have achieved an expression in terms of  $r, u, I, \Lambda$ , which is not the full set of Kepler elements yet. This partial results has to be complemented by the following transformation:

$$\frac{1}{r^{l+1}} e^{ik\nu} = \frac{1}{a^{l+1}} \sum_{q=-\infty}^{\infty} G_{lkq}(e) e^{i(k+q)M}, \quad (5.15)$$

which can be regarded as a Fourier transformation of the function  $e^{ik\nu}/r^{l+1}$ . The Fourier coefficients  $G_{lkq}(e)$  are called *eccentricity functions*. This transformation finalizes the



required form of the geopotential in terms of Kepler elements:

$$V(\mathbf{s}) = \frac{GM}{R} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{k=-l,2}^l \sum_{q=-\infty}^{\infty} \left(\frac{R_E}{a}\right)^{l+1} K_{lm} F_{lmk}(I) G_{lkq}(e) e^{i\psi_{mkq}} \quad (5.16a)$$

$$\text{with } \psi_{mkq} = k\omega + (k+q)M + m\Lambda \quad (5.16b)$$

The fourth summation over  $q$  runs in principle from  $-\infty$  to  $\infty$ . However, the eccentricity functions decay rapidly according to:

$$G_{lkq}(e) \sim \mathcal{O}(e^{|q|}).$$

Therefore, the  $q$ -summation can be limited for most geodetic satellites to  $|q| \leq 1$  or 2 at most. Note that the *metric* Kepler elements  $(a, e, I)$  appear in the upward continuation, eccentricity and inclination functions, whereas the *angular* Kepler elements define the angular variable  $\psi_{mkq}$ .

If the  $p$ -index is used for a Kaula-type of inclination function, the eccentricity function becomes  $G_{lpq}(e)$ . Moreover, the composite angle  $\psi_{mkq}$  turns into:

$$\psi_{lmpq} = (l-2p)\omega + (l-2p+q)M + m\Lambda.$$

The apparent dependence of  $\psi$  on the degree  $l$  is artificial.

**Real-valued expression.** If we return to real-valued coefficients and functions, the inclination functions need to become real too. Only the term  $i^{k-m}$  in (5.13) needs to be adapted. Since  $l$  and  $k$  have the same parity, due to  $P_{lk}(0) = 0$  for  $l-k$  odd, we can write:

$$i^{k-m} = i^{l-2p-m} = (-1)^p i^{l-m} = (-1)^{\frac{l-k}{2}} i^{l-m}.$$

The power of  $(-1)$  can be absorbed into the definition of a real-valued inclination function. The power of  $i$  needs to be taken care of by a case distinction between  $l-m$  even or odd and by a proper selection of either  $C_{lm}$  or  $S_{lm}$ . After some manipulations (5.16a) is recast into:

$$V(\mathbf{s}) = \frac{GM}{R} \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{k=-l,2}^l \sum_{q=-\infty}^{\infty} \left(\frac{R_E}{a}\right)^{l+1} F_{lmk}(I) G_{lkq}(e) S_{lmkq}(\omega, \Omega, M)$$

$$S_{lmkq}(\omega, \Omega, M) = \left[ \begin{array}{c} \left( \begin{array}{c} C_{lm} \\ -S_{lm} \end{array} \right) \cos \psi_{mkq} + \left( \begin{array}{c} S_{lm} \\ C_{lm} \end{array} \right) \sin \psi_{mkq} \end{array} \right]_{l-m \text{ odd}}^{l-m \text{ even}} \quad (5.17)$$

with the same definition of  $\psi_{mkq}$ . Again, one may use the  $p$ -index in order to have  $\sum_{p=0}^l$  as the 3<sup>rd</sup> summation. Also, recall that real-valued quantities use a slightly different normalization factor.

### 5.3 Lumped coefficient representation

Let us return to (5.14), i.e. the expression of the geopotential in terms Kepler elements before introducing the eccentricity functions. The part  $\exp(i(ku + m\Lambda))$  reminds of a 2D-Fourier series. The argument of latitude  $u$  and the longitude of the ascending node  $\Lambda$  attain values in the range  $[0; 2\pi)$ . Topologically, the product  $[0; 2\pi) \times [0; 2\pi)$  yields a torus, which is the proper domain of a 2D-Fourier series. Indeed the potential can be recast into a 2D-Fourier expression, if the following Fourier coefficients are introduced:

$$A_{mk}^V = \sum_{l=\max(|m|,|k|)}^{\infty} H_{lmk}^V K_{lm}, \quad (5.18a)$$

$$\text{with } H_{lmk}^V = \frac{GM}{R} \left( \frac{R_E}{r} \right)^{l+1} F_{lmk}(I). \quad (5.18b)$$

With these quantities, the potential reduces to the series:

$$V(u, \Lambda) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A_{mk}^V e^{i\psi_{mk}}, \quad (5.18c)$$

$$\psi_{mk} = ku + m\Lambda. \quad (5.18d)$$

Just like (5.14), the above equations are valid for any orbit. They are not necessarily restricted to circular orbits. The 2D-Fourier expression (5.18) makes only sense, though, on an orbit with constant  $I$  and  $r$ . This is the concept of a *nominal orbit*. Only then do the  $H_{lmk}^V$  and correspondingly the Fourier coefficients  $A_{mk}^V$  become time independent.

The Fourier coefficients  $A_{mk}^V$  are usually referred to in literature as *lumped coefficients*, since they are a sum (over degree  $l$ ). All potential coefficients  $K_{lm}$  of a specific order  $m$  are lumped in a linear way into  $A_{mk}^V$ . The coefficients  $H_{lmk}^V$  are denoted *transfer coefficients* here. They transfer the spherical harmonic spectrum into a Fourier spectrum. They are also known as *sensitivity* and *influence* coefficients.

Both  $A_{mk}$  and  $H_{lmk}$  are labelled by a super index  $V$ , referring to the geopotential  $V$ . In the next section, we will see that the same formulation can be applied to any functional of the geopotential. Only the transfer coefficients is specific to a particular functional.

**Remark 5.4 (Lumped coefficients)** *The word lumped merely indicates an accumulation of numbers, e.g. here a linear combination of potential coefficients over degree  $l$ , in general. Nevertheless a host of definitions and notations of lumped coefficients exists.*

An early reference where lumped coefficients are determined and discussed, is (Gooding, 1971). See (Klokočník, Kostelecký & Li, 1990) for a list of lumped coefficients from

several resonant orbit perturbations. Also in (Heiskanen & Moritz, 1967) lumped coefficients are discussed; zonal lumped coefficients, to be precise, that include non-linearities.

## 5.4 Pocket guide of dynamic satellite geodesy

Not only the potential, but also its functionals can be represented by a 2D-Fourier series, similar to (5.18). For  $f^\#$ , in which the label  $\#$  represents a specific functional, the spectral decomposition is:

$$f^\# = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A_{mk}^\# e^{i(ku + m\Lambda)}, \text{ with} \quad (5.19a)$$

$$A_{mk}^\# = \sum_{l=\max(|m|,|k|)}^{\infty} H_{lmk}^\# K_{lm}. \quad (5.19b)$$

By means of the above equations, a linear observation model is established, that links functionals of the geopotential to the fundamental parameters, the spherical harmonic coefficients. The link is in the spectral domain. The elementary building blocks in this approach are transfer coefficients, similar to (5.18b). The linear model provides a basic tool for gravity field analyses. E.g. the recovery capability of future satellite missions can be assessed, or the influence of gravity field uncertainties on other functionals.

**Pocket guide vs. Meissl scheme** A collection of transfer coefficients  $H_{lmk}^\#$  for all relevant functionals—observable or not—will be denoted as a *pocket guide* (PG) to dynamic satellite geodesy. Such a PG reminds of the *Meissl scheme*, cf. (Rummel & Van Gelderen, 1995), which presents the spectral characteristics of the first and second order derivatives of the geopotential. This scheme enables to link observable gravity-related quantities to the geopotential field. A major difference between the PG and the Meissl-scheme is, that the former links SH coefficients to Fourier coefficients, whereas the latter stays in one spectral domain, either spherical harmonic or Fourier. Consequently, the transfer coefficients do not solely depend on SH degree  $l$ . In general, the spherical harmonic orders  $m$  and  $k$  are involved as well. The transfer coefficients can not be considered as eigenvalues of a linear operator, representing the observable, as in the case of the Meissl-scheme.

## 5.5 Derivatives of the geopotential

In this section the transfer coefficients of the first and second spatial derivatives of the potential are derived in the local satellite frame:  $x$  quasi along-track,  $y$  cross-track and

$z$  radial.

Since the satellite is in free fall, the gradient of the potential,  $\nabla V$ , is not an observable functional. Nevertheless, the gradient vector—and consequently its transfer coefficients—are highly relevant. They supply the force function to the dynamic equations. In particular, with the derivatives in the satellite frame, the resulting gradient vector can directly be used in Gauss-type equations of motion. In 5.5.1 the transfer coefficients of all gradient components will be derived.

*Gravity gradiometry* is the measurement of the gradient of the gravity vector, which is a gradient by itself. The gradient of a gradient of a potential is a matrix or tensor of second derivatives. The gravity gradient tensor is also referred to as *Hesse matrix* in mathematics or *Marussi tensor* in physical geodesy. In 5.5.2, the transfer coefficients of all tensor components will be derived, also in the local satellite frame.

### 5.5.1 First derivatives: gravitational attraction

Before applying the gradient operator  $\nabla = [\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z}]^T = [\partial_x \ \partial_y \ \partial_z]^T$  to the geopotential expression (5.14) or to (5.18a)–(5.18d), it is recalled that in the rotated geocentric system  $u$  plays the role of longitude,  $\theta'$  that of co-latitude (although its nominal value is fixed at  $\frac{1}{2}\pi$ ) and  $r$  is the radial coordinate of course. Thus the gradient operator in the satellite frame becomes:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \frac{\partial}{\partial u} \\ -\frac{1}{r} \frac{\partial}{\partial \theta'} \\ \frac{\partial}{\partial r} \end{pmatrix}.$$

Let the potential be written as  $V = \sum_{lmk} V_{lmk}$ . Then the mechanism for deriving transfer coefficients is explained for the  $x$  and  $z$  components:

$$\begin{aligned} \partial_x V_{lmk} &= \frac{1}{r} \frac{\partial V_{lmk}}{\partial u} = \frac{1}{r} \frac{\partial V_{lmk}}{\partial e^{i\psi_{mk}}} \frac{\partial e^{i\psi_{mk}}}{\partial u} = \frac{ik}{r} V_{lmk}, \\ \partial_z V_{lmk} &= \frac{\partial V_{lmk}}{\partial r} = \frac{\partial V_{lmk}}{\partial (R/r)^{l+1}} \frac{\partial (R/r)^{l+1}}{\partial r} = -\frac{l+1}{r} V_{lmk}. \end{aligned}$$

So the along-track component of the gradient,  $\partial_x V$ , will be characterized by a term  $ik/r$ , and the radial derivative by the usual  $-(l+1)/r$ .

**Cross-track derivative.** The cross-track component requires special attention. The  $\theta'$ -coordinate is hidden in the inclination function  $F_{lmk}(I)$ , (5.13). It is therefore convenient to introduce a cross-track derivative of the inclination function, denoted as  $F_{lmk}^*(I)$ , cf. (Sneeuw, 1992):

$$F_{lmk}^*(I) = -\frac{\partial F_{lmk}(I)}{\partial \theta'} = i^{k-m+2} d_{lmk}(I) \left. \frac{dP_{lk}(\cos \theta')}{d\theta'} \right|_{\theta'=\pi/2}.$$

With the parameter  $t = \cos \theta$  the derivatives are:  $\frac{dP_{lk}(t)}{dt} = -\frac{dP_{lk}(\cos \theta)}{\sin \theta d\theta}$ . At the equator ( $\theta = \pi/2$ , or  $t = 0$ ) no confusion about the  $\sin \theta$  factor can arise. Let the derivative with respect to  $t$  be simply called  $P'_{lk}(0)$ , then the *cross-track inclination function* is defined as:

$$F_{lmk}^*(I) = i^{k-m} d_{lmk}(I) P'_{lk}(0). \quad (5.20)$$

When applying recursions of derivatives of Legendre functions, e.g. (Ilk, 1983), to the equator, one obtains:

$$(1-t^2) \frac{dP_{lk}(t)}{dt} = \sqrt{1-t^2} P_{l,k+1}(t) - kt P_{lk}(t) \xrightarrow{t=0} P'_{lk}(0) = P_{l,k+1}(0). \quad (5.21)$$

So the derivative  $P'_{lk}$  will be an even function for  $l-k$  odd and an odd one for  $l-k$  even. Thus the cross-track inclination functions will vanish for  $l-k$  even. This would allow the introduction of a Kaula-like cross-track inclination function  $F_{lmp}^*(I)$ .

**Alternative cross-track derivatives.** Other approaches, circumventing the introduction of  $F_{lmk}^*(I)$ , exist. Colombo (1986) suggested as cross-track derivative the expression

$$\frac{\partial}{\partial y} = \frac{1}{r \sin u} \frac{\partial}{\partial I},$$

which shows singularities in  $u$ . See also (Betti & Sans, 1989; Rummel et al., 1993, A.3.2). Depending on coordinate choice, better worked out in (Koop, 1993) or (Balmino, Schrama & Sneeuw, 1996), other expressions can be derived, e.g. the following singular one:

$$\frac{\partial}{\partial y} = \frac{1}{r \cos u \sin I} \left( \cos I \frac{\partial}{\partial u} - \frac{\partial}{\partial \Lambda} \right).$$

By multiplying the former by  $\sin^2 u$ , the latter by  $\cos^2 u$  and adding the result, Schrama (1989) derived the *regular* expression:

$$\frac{\partial}{\partial y} = \frac{1}{r} \left[ \sin u \frac{\partial}{\partial I} + \frac{\cos u}{\sin I} \left( \cos I \frac{\partial}{\partial u} - \frac{\partial}{\partial \Lambda} \right) \right],$$

which leads to a corresponding cross-track inclination function:

$$F_{lmk}^*(I) = \frac{1}{2} \left[ \frac{(k-1)\cos I - m}{\sin I} \right] \bar{F}_{lm,k-1}(I) - \frac{1}{2} \bar{F}'_{lm,k-1}(I) + \frac{1}{2} \left[ \frac{(k+1)\cos I - m}{\sin I} \right] \bar{F}_{lm,k+1}(I) + \frac{1}{2} \bar{F}'_{lm,k+1}(I), \quad (5.22)$$

where the primes denote differentiation with respect to inclination  $I$ . Although numerical equivalence between the real version of (5.20) and (5.22) could be verified, it was proven analytically in (Balmino et al., 1996) that this last expression consists in fact of a twofold definition:

$$F_{lmk}^*(I) = \left[ \frac{(k-1)\cos I - m}{\sin I} \right] \bar{F}_{lm,k-1}(I) - \bar{F}'_{lm,k-1}(I), \quad (5.23a)$$

$$F_{lmk}^*(I) = \left[ \frac{(k+1)\cos I - m}{\sin I} \right] \bar{F}_{lm,k+1}(I) + \bar{F}'_{lm,k+1}(I). \quad (5.23b)$$

**In summary,** the spectral characteristics of the gradient operator in the local satellite frame are given by the following transfer coefficients:

$$\partial_x \quad : \quad H_{lmk}^x = \frac{GM}{R^2} \left( \frac{R_E}{r} \right)^{l+2} \quad [ik] \quad F_{lmk}(I) \quad (5.24a)$$

$$\partial_y \quad : \quad H_{lmk}^y = \frac{GM}{R^2} \left( \frac{R_E}{r} \right)^{l+2} \quad [1] \quad F_{lmk}^*(I) \quad (5.24b)$$

$$\partial_z \quad : \quad H_{lmk}^z = \frac{GM}{R^2} \left( \frac{R_E}{r} \right)^{l+2} \quad [-(l+1)] \quad F_{lmk}(I) \quad (5.24c)$$

**Remark 5.5 (Nomenclature)** *The different parts in these transfer coefficients will be denoted in the sequel as dimensioning term containing  $(GM, R)$ , upward continuation term (a power of  $R/r$ ), specific transfer and inclination function part. Especially the specific transfer is characteristic for a given observable.*

According to this nomenclature, the specific transfer of the potential is 1, cf. equation (5.18b). Both  $H_{lmk}^x$  and  $H_{lmk}^z$  show a transfer of  $\mathcal{O}(l, k)$  which is specific to first derivatives in general. Higher frequencies are amplified. The same holds true for  $H_{lmk}^y$ , though hidden in  $F_{lmk}^*(I)$ . Equations (5.23a) indicate already that  $F_{lmk}^*(I) \sim \mathcal{O}(l, k) \times F_{lmk}(I)$ . This becomes clearer for the second cross-track derivative, cf. next section. Note also that only the radial derivative is isotropic, i.e. only depends on degree  $l$ . Its specific transfer is invariant under rotations of the coordinate system like (5.11b). This is not the case for  $V_x$  and  $V_y$ , when considered as scalar fields.

### 5.5.2 Second derivatives: the gravity gradient tensor

In contrast to the first derivatives, the second derivatives of the geopotential field are observable quantities. The observation of these is called *gravity gradiometry*, whose technical realization is described e.g. in (Rummel, 1986a). For a historical overview of measurement principles and proposed satellite gradiometer missions, refer to (Forward, 1973; Rummel, 1986b).

The gravity gradient tensor of second derivatives reads:

$$\mathbf{V} = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix}. \quad (5.25)$$

The sub-indices denote differentiation with respect to the specified coordinates. The tensor  $\mathbf{V}$  is symmetric. Due to Laplace's equation  $\Delta V = V_{xx} + V_{yy} + V_{zz} = 0$ , it is also trace-free. In local spherical coordinates  $(r, u, \theta')$  the tensor can be expressed as, e.g. (Koop, 1993, eqn. (3.10)):

$$\mathbf{V} = \begin{pmatrix} \frac{1}{r^2}V_{uu} + \frac{1}{r}V_r & -\frac{1}{r^2}V_{\theta'u} & \frac{1}{r}V_{ur} - \frac{1}{r^2}V_u \\ & \frac{1}{r^2}V_{\theta'\theta'} + \frac{1}{r}V_r & -\frac{1}{r}V_{\theta'r} + \frac{1}{r^2}V_{\theta'} \\ \text{symm.} & & V_{rr} \end{pmatrix}. \quad (5.26)$$

Again, use has been made of the fact that the satellite is always on the rotated equator  $\theta' = \frac{1}{2}\pi$ . With Laplace's equation one can avoid a second differentiation with respect to the  $\theta'$ -coordinate by writing:

$$V_{yy} = -V_{xx} - V_{zz} = -\frac{1}{r^2}V_{uu} - \frac{1}{r}V_r - V_{rr}.$$

As usual, the purely radial derivative is the simplest one. It is spectrally characterized by:  $(l+1)(l+2)/r^2$ . The operator  $\partial_{xx}$  will return the term:  $-[k^2 + (l+1)]/r^2$ . The second cross-track derivative  $\partial_{yy}$  thus gives with Laplace  $[k^2 + (l+1) - (l+1)(l+2)]/r^2 = [k^2 - (l+1)^2]/r^2$ . The *spectral transfer* for  $\partial_{xz}$  becomes:  $[-ik(l+1) - ik]/r^2 = -ik(l+2)/r^2$ . The components  $V_{xy}$  and  $V_{yz}$  make use of  $\partial_{\theta'}$ , which requires the use of  $F_{lmk}^*(I)$  again. Starting from the expression for  $V_y$ , one further  $ik/r$ -term is required to obtain  $V_{xy}$ . For  $V_{yz}$  one needs an extra  $[-(l+1) - 1]/r = -(l+2)/r$ . The full set of transfer coefficients, describing the single components of the gravity gradient tensor, is thus given by:

$$\partial_{xx} \quad : \quad H_{lmk}^{xx} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [-(k^2 + l + 1)] F_{lmk}(I) \quad (5.27a)$$

$$\partial_{yy} \quad : \quad H_{lmk}^{yy} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [k^2 - (l + 1)^2] F_{lmk}(I) \quad (5.27b)$$

$$\partial_{zz} \quad : \quad H_{lmk}^{zz} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [(l+1)(l+2)] F_{lmk}(I) \quad (5.27c)$$

$$\partial_{xy} \quad : \quad H_{lmk}^{xy} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [ik] F_{lmk}^*(I) \quad (5.27d)$$

$$\partial_{xz} \quad : \quad H_{lmk}^{xz} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [-ik(l+2)] F_{lmk}(I) \quad (5.27e)$$

$$\partial_{yz} \quad : \quad H_{lmk}^{yz} = \frac{GM}{R^3} \left( \frac{R_E}{r} \right)^{l+3} [-(l+2)] F_{lmk}^*(I) \quad (5.27f)$$

The specific transfer is of order  $\mathcal{O}(l^2, lk, k^2)$ , as can be expected for second derivatives. This is also true for  $H_{lmk}^{xy}$  and  $H_{lmk}^{yz}$ , that make use of  $F_{lmk}^*(I)$ . Again, the purely radial derivative is the only isotropic component. Adding the specific transfers of the diagonal components yields the Laplace equation in the spectral domain:

$$-(k^2 + l + 1) + k^2 - (l + 1)^2 + (l + 1)(l + 2) = 0.$$

**Alternative cross-track gravity gradient.** An alternative derivation of  $V_{yy}$  could have been obtained directly, i.e. without the Laplace equation, by a second cross-track differentiation. A new inclination function, say  $F_{lmk}^{**}(I)$  is required, defined as:

$$F_{lmk}^{**}(I) = \frac{\partial^2 F_{lmk}(I)}{\partial \theta'^2} = i^{k-m} d_{lmk}(I) \bar{P}_{lk}''(0).$$

From known recursions (Ilk, 1983), we have for the second latitudinal derivative of the unnormalized Legendre function at the equator:

$$P_{lk}''(0) = [k^2 - l(l+1)] P_{lk}(0).$$

A normalized version of this expression must be inserted in the definition of  $F_{lmk}^{**}(I)$  above, yielding the specific transfer  $[k^2 - l(l+1)]$  of the second cross-track derivative  $V_{\theta'\theta'}$ . Since  $V_{yy} = V_{\theta'\theta'}/r^2 + V_r/r$  one ends up with exactly the same transfer, as derived above with the Laplace equation, namely  $[k^2 - (l+1)^2]/r^2$ . Moreover, it demonstrates again that  $F_{lmk}^*(I)$  is of order  $\mathcal{O}(l, k)$ , since the second cross-track derivative has a transfer of  $\mathcal{O}(l^2, lk, k^2)$ .

**Space-stable gradiometry.** The transfer coefficients (5.27) pertain to tensor components in the local satellite frame. Especially for local-level orientations, such as Earth-pointing, these expressions are useful. In principle any other orientation can be deduced from them, since a tensor  $\mathbf{V}$  is transformed into another coordinate system by:

$$\mathbf{V}' = \mathbf{R}\mathbf{V}\mathbf{R}^\top,$$



cf. (Koop, 1993), in which  $R$  is the rotation matrix between the two systems. For instance the rotation sequence

$$R = R_z(-\Lambda)R_x(-I)R_z(-u),$$

which is the inverse of the rotations from 5.2, may be used to transform the gravity gradient tensor back into an Earth-fixed reference frame. Note, however, that the angles  $u$  and  $\Lambda$  are time-dependent. The derivation of transfer functions becomes cumbersome. An alternative approach, based on the work of Hotine (1969), is followed by Ilk (1983) and Bettadpur (1991, 1995).

## 6 Gravitational orbit perturbations

We are now able to write down the equations of motion of a satellite in a gravitational field. To that end we need to take the partial derivatives of the gravitational potential (5.16a) to all Kepler elements and combine them according to the LPE (??). The first step we take in 6.1 is to solve the LPE for the main effect, that is the secular orbit change due to the flattening of the Earth. In the subsequent section 6.2 we will derive the remaining gravitational orbit perturbations from linear perturbation theory (LPT). In 6.3 we will discuss the orbit perturbation spectrum and related aspects like resonance.

### 6.1 The $J_2$ secular reference orbit

The main deviation from a central gravitational field  $GM/r$  is caused by the dynamic flattening of the Earth. In the GRS80 normal field the flattening is represented by the dimensionless constant  $J_2 = 1.08263 \cdot 10^{-3}$ . For actual gravity fields, it is represented by the spherical harmonic coefficient  $C_{2,0} = K_{2,0} \approx -J_2$ . To be precise, these are non-normalized coefficients. A division by  $\sqrt{5}$  would normalize them.

The gravitational field produced by  $K_{2,0}$  reads:

$$V_{2,0}(\mathbf{s}) = \frac{GM}{R_E} \left( \frac{R_E}{a} \right)^3 K_{2,0} \sum_{k=-2,2}^2 \sum_{q=-\infty}^{\infty} F_{2,0,k}(I) G_{2,k,q}(e) e^{i[k\omega + (k+q)M]}.$$

It can be expected that periodic excitations give mainly rise to periodic perturbations. Thus the main perturbation can be expected from the zero-frequency term with  $k = q = 0$ :

$$\begin{aligned} R = V_{2,0,0,0}(a, e, I) &= \frac{GM}{R_E} \left( \frac{R_E}{a} \right)^3 K_{2,0} F_{2,0,0}(I) G_{2,0,0}(e) \\ &= \frac{GM}{R_E} \left( \frac{R_E}{a} \right)^3 C_{2,0} \left( \frac{3}{4} \sin^2 I - \frac{1}{2} \right) (1 - e^2)^{-\frac{3}{2}} \end{aligned}$$

The LPE require the partial derivatives of this expression. The partial derivatives w.r.t. the angular variables are all zero. Only the following remain:

$$\frac{\partial R}{\partial a} = -3GM \frac{R_E^2}{a^4} C_{2,0} \left( \frac{3}{4} \sin^2 I - \frac{1}{2} \right) (1 - e^2)^{-\frac{3}{2}} \quad (6.1a)$$

$$\frac{\partial R}{\partial I} = GM \frac{R_E^2}{a^3} C_{2,0} \frac{3}{2} \sin I \cos I (1 - e^2)^{-\frac{3}{2}} \quad (6.1b)$$

$$\frac{\partial R}{\partial e} = 3eGM \frac{R_E^2}{a^3} C_{2,0} \left( \frac{3}{4} \sin^2 I - \frac{1}{2} \right) (1 - e^2)^{-\frac{5}{2}} \quad (6.1c)$$

These partial derivatives are to be inserted in (??). Substituting  $GM = n^2 a^3$  and performing the necessary simplifications will yield the LPE for secular orbital motion due to the flattening of the Earth:

$$\dot{a} = 0 \quad (6.2a)$$

$$\dot{e} = 0 \quad (6.2b)$$

$$\dot{I} = 0 \quad (6.2c)$$

$$\dot{\omega} = \frac{3}{4} n C_{2,0} \frac{1}{(1 - e^2)^2} \left( \frac{R_E}{a} \right)^2 (1 - 5 \cos^2 I) \quad (6.2d)$$

$$\dot{\Omega} = \frac{3}{2} n C_{2,0} \frac{1}{(1 - e^2)^2} \left( \frac{R_E}{a} \right)^2 \cos I \quad (6.2e)$$

$$\dot{M} = n - \frac{3}{4} n C_{2,0} \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left( \frac{R_E}{a} \right)^2 (3 \cos^2 I - 1) \quad (6.2f)$$

The first three of these equations are trivially solved:  $a$ ,  $e$  and  $I$  are constant. The orbit does not change its size and shape under the influence of the Earth's flattening. Nor does the inclination change. With the metric Kepler elements constant, the right hand sides of the remaining three LPE become constant too. The full set of differential equations (6.2a) is easily integrable to:

$$a(t) = a_0 \quad (6.3a)$$

$$e(t) = e_0 \quad (6.3b)$$

$$I(t) = I_0 \quad (6.3c)$$

$$\omega(t) = \omega(t_0) + \dot{\omega}(t - t_0), \quad (6.3d)$$

$$\Omega(t) = \Omega(t_0) + \dot{\Omega}(t - t_0), \quad (6.3e)$$

$$M(t) = M(t_0) + \dot{M}(t - t_0), \quad (6.3f)$$

with the above indicated rates. The nodal line will precess at a constant rate  $\dot{\Omega}$ . Also the perigee will precess linearly in time. Moreover, the flattened Earth causes the mean anomaly to accelerate (or decelerate). An orbit with these constant angular rates is

called *secular*. In summary, the secular  $J_2$ -orbit is characterized by:

$$\text{secular } J_2 \text{ orbit: } \boxed{\begin{array}{l} a, e, I \text{ constant} \\ \dot{\omega}, \dot{\Omega}, \dot{M} \text{ constant} \end{array}} \quad (6.4)$$

**Perigee precession.** The perigee precession rate depends on the inclination. It can be made to zero if  $\cos^2 I = \frac{1}{5}$ , resulting in the *critical inclinations*  $I = 63^\circ.4$  or  $I = 116^\circ.6$ . For lower inclinations—orbital plane closer to the equator—the perigee precession rate becomes positive: both  $(1 - 5 \cos^2 I)$  and  $C_{2,0}$  are negative. For higher inclinations—orbital plane closer to the poles— $\dot{\omega}$  is negative.

The Russian communication satellite system Molniya makes a clever use of this property. Molniya satellites are in a highly eccentric orbit ( $e \approx 0.74$ ). After sweeping through perigee, they will move slowly and be visible for a long time. To ensure that this occurs over Russia, or over the Northern hemisphere in general, the perigee must be fixed over the Southern hemisphere at  $\omega = 270^\circ$ . This is done by choosing an inclination of  $63^\circ.4$ .

Perigee precession will also occur for equatorial orbits, or, in a heliocentric setting, for ecliptical orbits. Thus the relativistic perigee advance of Mercury’s orbit around the sun, may be obscured by an inadequately known gravitational flattening of the Sun.

**Nodal precession.** The nodal precession is proportional to  $\cos I$ . Thus, the plane of polar orbits will not change. This can be expected, since the rotationally symmetric flattened Earth does not exert a gravitational torque on a polar orbit. For prograde orbits, the nodes will move clockwise ( $\dot{\Omega} < 0$ ), whereas  $\dot{\Omega} > 0$  for retrograde orbits.

**Mean motion change.** Similarly, the mean motion change due to the Earth’s flattening is proportional to  $(3 \cos^2 I - 1)$ . On orbits with an inclination lower than  $54^\circ.7$  or higher than  $125^\circ.3$  the satellite will actually move faster than the mean motion  $n$ . In between these inclinations, the satellite is held back by the gravitational torque.

**Remark 6.1 (Numerical example)** For a satellite at about 750 km height, following a near-circular orbit (e.g.  $e = 0.01$ ), the angular rates (6.2a) typically become:

$$\begin{aligned} \dot{\omega} &\approx 3^\circ.35 \left( 5 \cos^2 I - 1 \right) \text{ per day} \\ \dot{\Omega} &\approx -6^\circ.7 \cos I \text{ per day} \\ \dot{M} &\approx 14.4 + \frac{3^\circ.35}{360^\circ} \left( 3 \cos^2 I - 1 \right) \text{ revolutions per day} \end{aligned}$$

## 6.2 Periodic gravity perturbations in linear approach

With the main orbit perturbation described by the  $J_2$  secular orbit, we will now derive periodic orbit perturbations due to the remaining spherical harmonic contributions  $V_{lmkq}$ , including the periodic  $J_2$  effects  $V_{2,0,k,q}$ . Since the LPE will be non-linear we will apply *linear perturbation theory* (LPT). The algorithm is as follows:

- Take the partial derivatives of (5.16a) to the Kepler elements,
- Insert the partial derivatives into the LPE (??),
- Evaluate the right hand side of the resulting non-linear equations on the  $J_2$  reference orbit, thus leading to a linear system,
- Replace the integration to time by integration to the angular variable  $\psi_{mkq}$ .

The resulting LPT solution is an *approximation* to the real orbit perturbations, because of the linearization on the reference orbit. In principle, the LPT solution might be inserted again into the right side of the LPE. The method of *successive approximation* would lead to higher approximations. This process is extremely laborious, though.

**Partial derivatives.** In the following we will abbreviate  $F_{lmk}(I)$  into  $F$  and  $G_{lkq}(e)$  into  $G$ . Primes will denote derivatives of the functions towards their argument. We will also recast the power of the upward continuation in (5.16a) by adjusting the dimensioning factor.

$$\begin{aligned}\frac{\partial V}{\partial a} &= \frac{GM}{a^2} \sum_{lmkq} [-(l+1)] \left(\frac{R_E}{a}\right)^l FGK_{lm} e^{i\psi_{mkq}} \\ \frac{\partial V}{\partial e} &= \frac{GM}{a} \sum_{lmkq} \left(\frac{R_E}{a}\right)^l FG'K_{lm} e^{i\psi_{mkq}} \\ \frac{\partial V}{\partial I} &= \frac{GM}{a} \sum_{lmkq} \left(\frac{R_E}{a}\right)^l F'GK_{lm} e^{i\psi_{mkq}} \\ \frac{\partial V}{\partial \omega} &= \frac{GM}{a} \sum_{lmkq} \left(\frac{R_E}{a}\right)^l FG[ik]K_{lm} e^{i\psi_{mkq}} \\ \frac{\partial V}{\partial \Omega} &= \frac{GM}{a} \sum_{lmkq} \left(\frac{R_E}{a}\right)^l FG[im]K_{lm} e^{i\psi_{mkq}} \\ \frac{\partial V}{\partial M} &= \frac{GM}{a} \sum_{lmkq} \left(\frac{R_E}{a}\right)^l FG[i(k+q)]K_{lm} e^{i\psi_{mkq}}\end{aligned}$$

**Insertion into LPE.** Collecting all derivatives, combining them into (??) and simplifying some factors using  $GM = n^2 a^3$  leads to the following set of Lagrange Planetary Equations:

$$\dot{a} = 2na \sum_{lmkq} \left( \frac{R_E}{a} \right)^l FGK_{lm} [k + q] i e^{i\psi_{mkq}} \quad (6.5a)$$

$$\dot{e} = \frac{n}{e} \sum_{lmkq} \left( \frac{R_E}{a} \right)^l FGK_{lm} \left[ (1 - e^2)(k + q) - \sqrt{1 - e^2}k \right] i e^{i\psi_{mkq}} \quad (6.5b)$$

$$\dot{I} = \frac{n}{\sin I \sqrt{1 - e^2}} \sum_{lmkq} \left( \frac{R_E}{a} \right)^l FGK_{lm} [k \cos I - m] i e^{i\psi_{mkq}} \quad (6.5c)$$

$$\dot{\omega} = n \sum_{lmkq} \left( \frac{R_E}{a} \right)^l \left[ F \frac{\sqrt{1 - e^2}}{e} G' - F' \frac{\cot I}{\sqrt{1 - e^2}} G \right] K_{lm} e^{i\psi_{mkq}} \quad (6.5d)$$

$$\dot{\Omega} = \frac{n}{\sin I \sqrt{1 - e^2}} \sum_{lmkq} \left( \frac{R_E}{a} \right)^l F' GK_{lm} e^{i\psi_{mkq}} \quad (6.5e)$$

$$\dot{M} = n - n \sum_{lmkq} \left( \frac{R_E}{a} \right)^l F \left[ \frac{1 - e^2}{e} G' - 2(l + 1)G \right] K_{lm} e^{i\psi_{mkq}} \quad (6.5f)$$

**Linear Perturbation Theory.** The above LPE (6.5a) are nonlinear ordinary differential equations. They can be solved in linear approximation. The  $J_2$  reference orbit (6.3a) is considered as the zero-order approximation, i.e. a *trajectory of Taylor points*. The remaining orbit perturbations are expected to be relatively small, that is, the real orbit is expected to oscillate around the reference orbit.

**Remark 6.2** *Naturally, all zonal coefficients will contribute a zero-frequency (DC) term with  $m = k = q = 0$ . Although they will be several orders of magnitude smaller than the  $J_2$ -effect, they will cause secular perturbations nevertheless. Thus it may be wise to incorporate the DC contributions from other zonal coefficients into the definition of the reference orbit, too.*

Now, the right hand side of (6.5a) is evaluated with with constant  $a$ ,  $e$  and  $I$ . At the same time, since the time  $t$  appears linearly in the complex exponentials, the time integration is replaced by an integration towards the angular variable  $\psi_{mkq}$ :

$$\int dt = \int \frac{dt}{d\psi} d\psi = \int \frac{1}{\dot{\psi}} d\psi = \frac{1}{\dot{\psi}} \int d\psi,$$

$$\text{with } \dot{\psi} = \dot{\psi}_{mkq} = k\dot{\omega} + (k+q)\dot{M} + m(\dot{\Lambda}).$$

The whole set of differential equations immediately becomes linear. A straightforward integration yields for each  $\{l, m, k, q\}$ -combination:

$$\Delta a_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} 2a \left( \frac{R_E}{a} \right)^l FGK_{lm} [k+q] e^{i\psi_{mkq}} \quad (6.6a)$$

$$\Delta e_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} \frac{1}{e} \left( \frac{R_E}{a} \right)^l FGK_{lm} \left[ (1-e^2)(k+q) - \sqrt{1-e^2}k \right] e^{i\psi_{mkq}} \quad (6.6b)$$

$$\Delta I_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} \frac{1}{\sin I \sqrt{1-e^2}} \left( \frac{R_E}{a} \right)^l FGK_{lm} [k \cos I - m] e^{i\psi_{mkq}} \quad (6.6c)$$

$$\Delta \omega_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} \left( \frac{R_E}{a} \right)^l \left[ F \frac{\sqrt{1-e^2}}{e} G' - F' \frac{\cot I}{\sqrt{1-e^2}} G \right] K_{lm} [-i] e^{i\psi_{mkq}} \quad (6.6d)$$

$$\Delta \Omega_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} \frac{1}{\sin I \sqrt{1-e^2}} \left( \frac{R_E}{a} \right)^l F' GK_{lm} [-i] e^{i\psi_{mkq}} \quad (6.6e)$$

$$\Delta M_{lmkq} = \frac{n}{\dot{\psi}_{mkq}} \left( \frac{R_E}{a} \right)^l F \left[ 2(l+1)G - \frac{1-e^2}{e} G' \right] K_{lm} [-i] e^{i\psi_{mkq}} \quad (6.6f)$$

The  $\Delta$ 's have to be understood as perturbations to the  $J_2$  reference orbit. In order to achieve the full orbit in linear perturbation theory we have to add the combined summations to the reference orbit :

$$\mathbf{s}(t) = \mathbf{s}_0 + \dot{\mathbf{s}}(t-t_0) + \sum_{l=2}^{\infty} \sum_{m=-l}^l \sum_{k=-l}^l \sum_{q=-\infty}^{\infty} \Delta \mathbf{s}_{lmkq}. \quad (6.7)$$

**Remark 6.3 (Linear orbit perturbations accuracy)** *The above orbit perturbation solution was achieved through linearization. The orbit perturbations (6.6a) are said to be linear. The main deviation from the zero-order solution, i.e. the  $J_2$  reference orbit, are the periodic effects due to  $C_{2,0}$ , which are of the order  $\mathcal{O}(J_2) = 10^{-3}$ . Thus, the zero-order solution achieves roughly a relative accuracy of  $10^{-3}$ . The linear solution has a relative precision of  $10^{-6}$ . The main approximation error is  $\mathcal{O}(J_2^2)$ .*

**Real-valued solutions.** To express the linear perturbations (6.6a) in terms of real-valued quantities, similar to (5.17), the following adaptations have to be made:

- The summation over  $m$  in (6.7) starts at  $m = 0$ .
- Combine terms with  $K_{lm} e^{i(ku + m\Lambda)}$  into  $S_{lmkq}$ , as defined in (5.17).

- Combine terms with  $K_{lm} [-i] e^{i\psi_{mkq}}$  into  $\bar{S}_{lmkq} = \int S_{lmkq} d\psi_{mkq}$ .
- Usually the  $k$ -summation is changed into a  $p$ -summation, including  $F_{lmp}(I)$  and  $G_{lpq}(e)$ .

Again it is seen that complex-valued expressions are far more compact. They are also more transparent. For instance, the terms  $[-i]$  that stem from the integration of  $e^{i\psi_{mkq}}$  are a simple phase-shift:  $-i = e^{-i\frac{\pi}{2}}$ . In the real-valued case, the integration from  $S_{lmkq}$  into  $\bar{S}_{lmkq}$  requires a more complicated interchange of  $C_{lm}$ ,  $S_{lm}$ , and cosines and sines.

### 6.3 The orbit perturbation spectrum

**Linear system.** In linear perturbation theory, the originally non-linear equations of motion (6.5a) were linearized at the  $J_2$  reference orbit. As a consequence (6.5a) became a set of linear ODE. One characteristic of linear systems is that an input forcing at a certain frequency causes an output disturbance at the same frequency. Indeed, if the Kepler elements are perturbed at a specific frequency  $\dot{\psi}_{mkq}$ , cf. the RHS of (6.5a), the resulting orbit perturbation (6.6a) will be at the same frequency.

In order to emphasize the spectral character of the orbit perturbations in a Fourier sense, we can go back to a lumped coefficient expression again as in (5.18). The main step is to turn the outer  $l$ -summation in (6.7) into an inner summation:

$$\sum_{l=2}^{\infty} \sum_{m=-l}^l \sum_{k=-l}^l \sum_{q=-\infty}^{\infty} \rightarrow \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{l=\max(|m|,|k|)}^{\infty},$$

perform the summation (i.e. lump) over the degree  $l$ , and collect the appropriate terms into a corresponding transfer coefficient. As an example, we can write for the perturbed semi-major axis:

$$\text{Fourier series:} \quad a(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{mkq}^a e^{i\dot{\psi}_{mkq}(t-t_0)}$$

$$\text{lumped coefficients:} \quad A_{mkq}^a = \sum_{l=\max(|m|,|k|)}^{\infty} H_{lmkq}^a K_{lm}$$

$$\text{transfer coefficients:} \quad H_{lmkq}^a = \frac{n}{\dot{\psi}_{mkq}} 2a \left( \frac{R_E}{a} \right)^l F_{lmk}(I) G_{lkq}(e) [k+q]$$

Which spherical harmonic coefficients contribute to a (lumped) Fourier coefficient  $A_{mkq}^{\#}$  at the frequency  $\dot{\psi}_{mkq}$ ? The frequency does not involve the degree  $l$ . Thus, if the  $\{mkq\}$ -combination is fixed, all spherical harmonic coefficients  $K_{lm}$  of that specific order  $m$  will



contribute, i.e. all degrees larger than  $m$ . Because of the vanishing inclination functions  $F_{lmk}(I)$  when  $(l - k)$  is odd, it will be either only the even or the odd degrees that contribute.

**Perturbation spectrum.** The perturbation frequencies  $\dot{\psi}_{mkq}$  are composite frequencies:

$$\dot{\psi}_{mkq} = k\dot{\omega} + (k + q)\dot{M} + m(\dot{\Omega} - \text{GAST}), \text{ or} \quad (6.8a)$$

$$= (k + q)(\dot{\omega} + \dot{M}) + m\dot{\Lambda} - q\dot{\omega}. \quad (6.8b)$$

The perigee drift  $\dot{\omega}$  and the nodal drift  $\dot{\Omega}$  are small: typically a few degrees per day. The frequency GAST is the daily rotation rate, i.e.  $360^\circ$  per day. Since this is far larger than the nodal rate (in absolute value), the *nodal daily rate*  $\dot{\Lambda} \approx -\text{GAST}$ . The frequency  $\dot{M}$  is the largest. For LEO satellites it is approximately 16 times faster than the daily rate.

The rewritten version (6.8b) is illustrative. The main frequency lines will be at an integer amount of  $(\dot{\omega} + \dot{M})$ , i.e. the orbital revolution frequency. These main peaks are interspersed with frequency lines  $m\dot{\Lambda}$ . On top of that, the orbit perturbations will be modulated by the apsidal frequency  $q\dot{\omega}$ .

For near-circular orbits the terms with  $q \neq 0$  will become small. The simplified perturbation spectrum reads:

$$\dot{\psi}_{mk} = k\dot{u} + m\dot{\Lambda}. \quad (6.8c)$$

**Resonance.** The linear orbit perturbations (6.6a) all contain a denominator  $\dot{\psi}_{mkq}$ . That means that the input forcing is greatly amplified for the low frequency spectrum. For an actual forcing at DC, i.e. the zero-frequency, the amplification would be infinite. This behaviour is called *resonance*, which is a common phenomenon in dynamical systems. If the dynamical system is excited close to the zero-frequency we have *near-resonance* or *shallow resonance*.

In case of exact resonance, the linear perturbation solution is invalid. A forcing at DC can simply not be represented by the type of oscillatory solutions as in (6.6a). Instead, a zero-frequency forcing will likely result in secular orbit perturbations similar to the  $C_{2,0}$  effect in 6.1. In case of near-resonance the linear perturbation theory does not necessarily break down, though care should be taken.

When does (near-)resonance occur? If we analyse the simplified frequency (6.8c), we can distinguish the following cases:

**zonals** With  $m = 0$  we have  $\dot{\psi}_{0,k} = k\dot{u}$ . Trivially, a zero frequency arises for  $k = 0$ , cf. 6.1. Thus, all even degree zonal coefficients will contribute.

***m*-dailies** As mentioned before, the nodal-daily frequency  $\dot{\Lambda}$  are nearly 16 times smaller than the orbital revolution rate. Thus, if  $k = 0$  the frequencies  $\dot{\psi}_{m,0} = m\dot{\Lambda}$  are close to zero, in particular for very low order  $m$ . Thus, low order coefficients give rise to near-resonance at a frequency of  $m$  cycles per day (CPD), hence the name *m*-dailies.

**repeat orbits** In general, resonance occurs if

$$\dot{\psi}_{mk} = 0 \Rightarrow k\dot{u} = -m\dot{\Lambda} \Rightarrow \frac{k}{m} = \frac{-\dot{\Lambda}}{\dot{u}} = \frac{T_u}{T_\Lambda},$$

in which  $T_u$  denotes the orbital revolution period and  $T_\Lambda$  is one nodal day.

Now the ratio  $\frac{k}{m}$  is an integer ratio. Thus the resonance condition can be met—i.e. we can find a suitable  $\{mk\}$ -combination—if the above periods  $T_u$  and  $T_\Lambda$  are in an integer ratio is well:

$$\frac{\dot{u}}{-\dot{\Lambda}} = \frac{T_\Lambda}{T_u} = \frac{\beta}{\alpha}. \quad (6.9)$$

This mathematical *commensurability* means geometrically a *repeat orbit*. After  $\beta$  revolutions exactly  $\alpha$  nodal days have passed. The integers  $\alpha$  and  $\beta$  have to be *relative primes*, i.e. they can not have a common divisor.

**Repeat orbits.** The repeat ratio  $\beta/\alpha$  will be close to 16 for LEO orbits, e.g. 49/3 or 31/2. For repeat orbits the simplified spectrum  $\dot{\psi}_{mk}$  can be simplified even further:

$$\dot{\psi}_{mk} = k\dot{u} + m\dot{\Lambda} = \dot{u} \left( k + m \frac{\dot{\Lambda}}{\dot{u}} \right) = \dot{u} \left( k - m \frac{\alpha}{\beta} \right) = \frac{\dot{u}}{\beta} (k\beta - m\alpha). \quad (6.10)$$

Since  $(k\beta - m\alpha)$  is solely composed of integers, we can map them onto a single integer  $n$ . The base frequency  $\dot{u}/\beta$  pertains to one full repeat period (of  $\alpha$  nodal days =  $\beta$  revolutions). With:

$$(k\beta - m\alpha) \mapsto n \quad \Rightarrow \quad \dot{\psi}_{mk} \mapsto \dot{\psi}_n = n \frac{\dot{u}}{\beta}.$$

Even if the repeat orbit condition (6.9) is not met, there will always be specific  $\{mk\}$ -combinations that, for the given  $\dot{u}$  and  $\dot{\Lambda}$ , give rise to the near-resonance situation  $\dot{\psi}_{mk} \approx 0$ . That will particularly occur when

$$m = \gamma \operatorname{int}\left(\frac{\beta}{\alpha}\right), \quad \gamma = 1, 2, \dots$$

As an example, suppose we have a  $49/3$  repeat ratio. Now take the situation  $k = 1$  and  $m = \text{int}(\frac{49}{3}) = 16$ . Then we get a near resonant frequency of

$$(k\beta - m\alpha) = (1 \cdot 49 - 16 \cdot 3) = 1 \implies \dot{\psi}_{16,1} = \frac{i}{49},$$

which is even smaller than the 1-daily near-resonance.

# 7 Modeling CHAMP, GRACE and GOCE observables

## 7.1 The Jacobi integral

The vis-viva equation described the total energy of a Kepler orbit. For real satellite orbits in a real and rotating Earth gravity field, the total energy will not be constant. Nevertheless, a constant of the motion can be found: the so-called Jacobi integral. It can be used to determine the gravity potential, more specifically the disturbing potential, as soon as the orbit is given in terms of position and velocity (and disturbing forces).

The Jacobi integral is a constant of the motion in a rotating frame. Here we will use an extended version that includes dissipative forces. The derivation of the Jacobi integral starts from the equation of motion in a rotating frame, whose rotation is prescribed by the vector  $\boldsymbol{\omega}$ . The kinematics in a rotating frame were derived in 4.1 in order to obtain the Hill equations of motion. In the following discussion, however,  $\boldsymbol{\omega}$  denotes the Earth's rotation vector, which is assumed to be constant in direction and rate:  $\boldsymbol{\omega} = (0 \ 0 \ \omega_E)^\top$ .

$$\ddot{\mathbf{r}} = \mathbf{f} + \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \dot{\boldsymbol{\omega}} \times \mathbf{r}, \quad (7.1a)$$

in which we have, at the right hand side, the dissipative force (per unit mass)  $\mathbf{f}$ , gravitational attraction  $\mathbf{g}$ , centrifugal, Coriolis and Euler acceleration, respectively. The gravitational attraction and the centrifugal acceleration are both conservative vector fields and can therefore be written as the gradient of the gravitational potential  $V$  and the centrifugal potential  $Z = \frac{1}{2}\omega^2(x^2 + y^2)$ , respectively. The vectors  $\mathbf{r}$ ,  $\dot{\mathbf{r}}$  and  $\ddot{\mathbf{r}}$  are positions, velocities and accelerations in the rotating frame. The Earth rotation  $\boldsymbol{\omega}$  is assumed to be constant, such that the Euler term cancels. Thus (7.1a) reduces to:

$$\ddot{\mathbf{r}} = \mathbf{f} + \nabla V + \nabla Z - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \quad (7.1b)$$

In 2.3 Newton's equation of motion was premultiplied by  $\dot{\mathbf{r}} \cdot \dots$  in order to establish energy conservation in the Kepler problem. We will now apply the same trick to (7.1a) in order to derive the Jacobi integral. Multiplying by the velocity  $\dot{\mathbf{r}}$  the part with the Coriolis acceleration will drop out:

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \dot{\mathbf{r}} \cdot \mathbf{f} + \dot{\mathbf{r}} \cdot (\nabla V + \nabla Z) - 2\dot{\mathbf{r}} \cdot (\boldsymbol{\omega} \times \dot{\mathbf{r}}) \quad (7.2)$$

$$= \dot{\mathbf{r}} \cdot \mathbf{f} + \frac{d(V + Z)}{dt} - \frac{\partial(V + Z)}{\partial t}$$

The latter step is due to the fact that the total time derivative of a potential is written as:

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial\Phi}{\partial t} = \dot{\mathbf{r}} \cdot \nabla\Phi + \frac{\partial\Phi}{\partial t} .$$

Because of the constant  $\omega$  the centrifugal potential has no explicit time derivative  $\partial Z/\partial t$ . Thus, upon integration, we are left with:

$$\int \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} dt = \int \left( \dot{\mathbf{r}} \cdot \mathbf{f} - \frac{\partial V}{\partial t} \right) dt + V + Z + c .$$

The integration constant  $c$  is called *Jacobi constant*. The left hand side is the kinetic energy (per unit mass)  $\frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ . The gravitational potential is split up in a normal (gravitational) part and a disturbance:  $V = U + T$ . Rearrangement gives:

$$T + c = E_{\text{kin}} - U - Z - \int \mathbf{f} \cdot d\mathbf{r} - \int \frac{\partial V}{\partial t} dt \quad (7.3)$$

Equation (7.3) is the basis for gravity field determination using the energy balance approach. At the left we have the unknown disturbing potential, up till an unknown constant. All terms at the right are determined from CHAMP data or existing models:

- $E_{\text{kin}}$  requires orbit velocities  $\dot{\mathbf{r}}$ ,
- $U$ , the normal gravitational potential, requires satellite positions  $\mathbf{r}$ ,
- $Z$ , the centrifugal potential at the satellite's location is also calculated from  $\mathbf{r}$ ,
- $\int \mathbf{f} \cdot d\mathbf{r}$  is the dissipated energy, which is an integral of CHAMP's accelerometer data  $\mathbf{f}$  along the orbit,
- $\int \partial_t V dt$  is the integral along the orbit of the gradient of time-variable potentials. It contains known sources (tides, 3<sup>rd</sup> bodies), that can be corrected in (7.3), and unknown gravity field changes.

## 7.2 Range, range rate and range acceleration

Whereas 4.1 described kinematics in a moving frame, we will now be concerned with the relative kinematics of a baseline between two satellites. We will discuss the intersatellite *range* and its first and second time derivative *range rate* and *range acceleration*. At the same time we will see how the baseline direction changes.

The baseline vector between two satellites is  $\boldsymbol{\rho}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ . The normalized baseline vector then becomes:

$$\mathbf{e}_{12} = \frac{\boldsymbol{\rho}_{12}}{|\boldsymbol{\rho}_{12}|} = \frac{\boldsymbol{\rho}_{12}}{\rho_{12}} .$$

**Remark 7.1 (notation)** *The quantities with indices 1 and 2 indicate differences between satellite 1 and 2. If it is clear from the context, the indices are dropped.*

**range**

The vectorial baseline is the scalar range times the unit vector in the direction of the baseline.

$$\boldsymbol{\rho} = \rho \mathbf{e} \quad \Longrightarrow \quad \rho = \boldsymbol{\rho} \cdot \mathbf{e} \quad (7.4)$$

**baseline**

The baseline direction  $\mathbf{e} = \boldsymbol{\rho}/\rho$  is a unit vector:

$$\mathbf{e} \cdot \mathbf{e} = 1 \quad \Longrightarrow \quad \mathbf{e} \cdot \dot{\mathbf{e}} = 0.$$

Hence, the time derivative  $\dot{\mathbf{e}}$  of the baseline vector is perpendicular to the baseline itself. The vector  $\dot{\mathbf{e}}$  itself is not a unit vector.

**range rate**

The scalar range rate  $\dot{\rho}$  is *not* the length of the relative velocity vector  $\dot{\boldsymbol{\rho}}$ . Instead it is the projection of the relative velocity onto the baseline.

$$\dot{\boldsymbol{\rho}} = \dot{\rho} \mathbf{e} + \rho \dot{\mathbf{e}} \quad \Longrightarrow \quad \dot{\boldsymbol{\rho}} \cdot \mathbf{e} = \underbrace{\dot{\rho} \mathbf{e} \cdot \mathbf{e}}_1 + \underbrace{\rho \dot{\mathbf{e}} \cdot \mathbf{e}}_0 \quad \Longrightarrow \quad \dot{\rho} = \dot{\boldsymbol{\rho}} \cdot \mathbf{e}. \quad (7.5)$$

An elementary way of writing this is:  $\boldsymbol{\rho} \cdot \dot{\boldsymbol{\rho}} = \rho \dot{\rho}$ .

**baseline (again)**

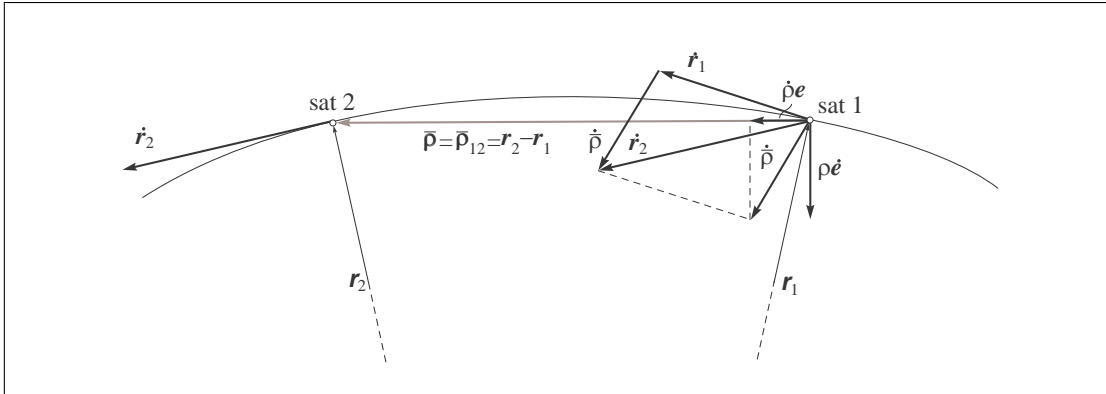
The above shows that the vector  $\dot{\mathbf{e}} = \frac{1}{\rho}(\dot{\boldsymbol{\rho}} - \dot{\rho} \mathbf{e})$  is obtained by subtracting the projection of  $\dot{\boldsymbol{\rho}}$  onto  $\mathbf{e}$  from the relative velocity vector  $\dot{\boldsymbol{\rho}}$  itself. The result will indeed be perpendicular to  $\mathbf{e}$ . This is nicely visualized in fig. 7.1. The perpendicular component of the relative velocity is called  $\mathbf{c}$ , for cross-track, defined as:

$$\mathbf{c} = \dot{\boldsymbol{\rho}} - \dot{\rho} \mathbf{e} = \rho \dot{\mathbf{e}}.$$

**range acceleration**

A further time differentiation yields

$$\begin{aligned} \ddot{\rho} &= \ddot{\boldsymbol{\rho}} \cdot \mathbf{e} + \dot{\boldsymbol{\rho}} \cdot \dot{\mathbf{e}} = \ddot{\boldsymbol{\rho}} \cdot \mathbf{e} + \dot{\boldsymbol{\rho}} \cdot \frac{1}{\rho}(\dot{\boldsymbol{\rho}} - \dot{\rho} \mathbf{e}) \\ \Longrightarrow \ddot{\rho} &= \ddot{\boldsymbol{\rho}} \cdot \mathbf{e} + \frac{1}{\rho}(\dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}} - \dot{\rho}^2). \end{aligned} \quad (7.6)$$



**Figure 7.1:** Position difference  $\bar{\rho}$ , range  $\rho$ , differential velocity  $\dot{\bar{\rho}}$  and range rate  $\dot{\rho}$

Using the perpendicular vector  $\mathbf{c}$  again, this is summarized into

$$\ddot{\bar{\rho}} = \ddot{\bar{\rho}} \cdot \mathbf{e} + \frac{1}{\rho} \mathbf{c} \cdot \mathbf{c}.$$

### 7.3 Spaceborne gravimetry

Inserting Newton's equations of motion—in differential mode—into the above range acceleration yields the basic SST equation for differential gravimetry

$$\begin{aligned} \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 &= \nabla V_2 - \nabla V_1 \\ \implies \ddot{\bar{\rho}}_{12} &= \nabla V_{12} \\ \implies \ddot{\bar{\rho}} &= \nabla V_{12} \cdot \mathbf{e} + \frac{1}{\rho} (\dot{\bar{\rho}} \cdot \dot{\bar{\rho}} - \dot{\rho}^2). \end{aligned}$$

The observable range acceleration  $\ddot{\bar{\rho}}$ , corrected for perpendicular velocity terms, equals the projection of the gradient difference onto the baseline. For low-low SST like GRACE the main problem is the size of the perpendicular velocity correction. It is larger than the range acceleration by orders of magnitude. It represents the (differential) centrifugal acceleration, projected onto the baseline.

But apart from numerical complications, the above formula demonstrates that in principle a GRACE-type observable can be considered as *spaceborne gravimetry*. Terrestrial gravimetry determines the length of the gravity vector, which is the projection of the gravity vector along the *plumb line*:  $g = \mathbf{g} \cdot \mathbf{e}_r = \nabla V \cdot \mathbf{e}_r$ . In spaceborne gravimetry we have something similar, although in differential mode: the projection of the gravity difference onto the baseline.

Lotlinie

## 7.4 GRACE-type gradiometry

The right hand side of the above equation was the difference of a gradient. When we divide this by the baseline length we obtain a gradient of a gradient, at least in linear approximation. Thus the range accelerometry can be interpreted as gravity gradiometry. Denoting the gravitational gradient tensor as  $\mathbf{V}$ , one obtains:

$$\frac{1}{\rho}\ddot{\rho} = \mathbf{e}^T \mathbf{V} \mathbf{e} + \frac{1}{\rho^2} (\dot{\rho} \cdot \dot{\rho} - \dot{\rho}^2) + \text{lin.error}.$$

By using a priori gravity field information the linearization error can be controlled, depending on maximum degree, baseline length and accuracy requirement.

In the gravity gradiometry literature, the observable tensor is usually expressed as:

$$\mathbf{\Gamma} = \mathbf{V} + \mathbf{\Omega}^2 + \dot{\mathbf{\Omega}},$$

with the latter two terms representing centrifugal and Euler acceleration differences. If we adopt a Hill frame ( $x$  quasi along-track,  $y$  cross-track and  $z$  radial) the range acceleration observable becomes along-track gradiometry:

$$\frac{1}{\rho}\ddot{\rho} = V_{xx} - (\omega_y^2 + \omega_z^2) + \text{lin.error}.$$

This shows again that the velocity correction terms represent the differential centrifugal acceleration:

$$\frac{1}{\rho^2}(\mathbf{c} \cdot \mathbf{c}) = -(\omega_y^2 + \omega_z^2).$$

For a LEO leader-follower configuration the nominal values of these terms are:

$$\omega_y = 1 \text{ CPR} = 0.18 \text{ mHz}$$

$$\omega_z = 0$$

Note that the centrifugal terms are independent of baseline length.

## 7.5 GOCE gradiometry



# 8 Perturbation theory

## 8.1 Classifications

### Klassifizierung 1

- gravitational: -  $K_{lm}$ , - 3rd body
- non gravitational: - drag, - albedo, - magnetic

### Klassifizierung 2

- body oder Volumenkraefte: - grav, - magnetic
- Oberflaechen: - drag, - albedo

### 8.1.1 Gravitationelle Stoerungen/Erdschwerefeld

$$\ddot{\mathbf{r}} = \nabla V = -\frac{GM}{r^3}\mathbf{r} + \nabla R$$

wobei  $\nabla R$  die Stoerkraft pro Masseinheit ( $\neq$  Beschleunigung).

$$V = \underbrace{\frac{GM}{r}}_{\text{Kepler- oder Zentralterm}} + \underbrace{R}_{\text{Stoerung}}$$

$$R = \frac{GM}{R} \sum_{l=2}^{\infty} \sum_{m=0}^l P_{lm}(\sin \phi) (C_{lm} \cos(m\lambda) + S_{lm} \sin(m\lambda)) \left(\frac{R}{r}\right)^{l+1}$$

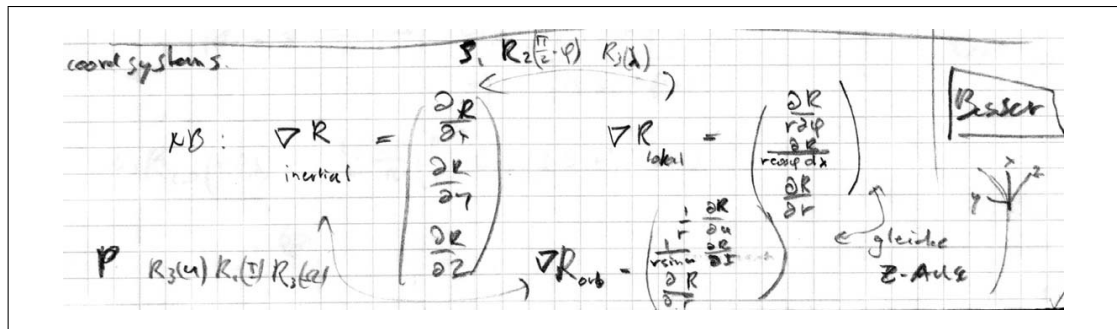


Figure 8.1

## 8.2 $J_2$ -perturbations

### 8.2.1 Qualitative assessment

### 8.2.2 Kreiseffekte ← Qualitative Betrachtung

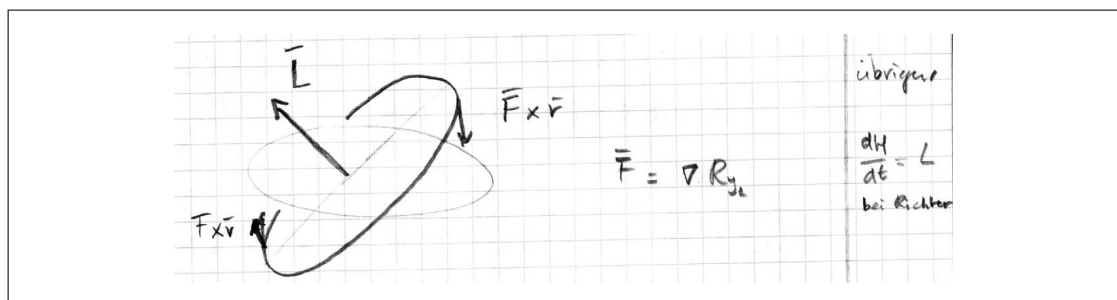


Figure 8.2

$\mathbf{T} = \mathbf{F} \times \mathbf{r}$  :  $\mathbf{T}$  = Drehmoment (Torque)  $\frac{d\mathbf{L}}{dt} = \mathbf{T}$  :  $\mathbf{L}$  = Drehimpuls (angular momentum)

“Drehmoment ändert den Drehimpuls”

z.B.:  $\mathbf{T} = \text{konstant} \implies \dot{\mathbf{L}} = \text{konstant} \implies$  Präzession (Bahnebene dreht sich, aber Neigung konstant.  $\implies \dot{\Omega}(\dot{I} = 0)$ )

Q: ist  $\mathbf{T}$  konstant?

- in Äquatorebene Null
- oben und unten maximal
- innen positiv (oder gleiche Richtung)

- 2 Mal pro Umlauf Variation
- Mittelwert  $> 0$

$$T = T_0 + A \cos(2nt) + B \sin(2nt)$$

### 8.2.3 Quantitative assessment

$$R_{2,0} = \frac{GM}{R} \left(\frac{R}{r}\right)^3 \frac{C_{2,0}}{-J_2} \frac{P_{2,0}(\sin \phi)}{\frac{1}{2}(3 \sin^2 \phi - 1)}, \quad C_{2,0} = -J_2$$

1.

$$R_{2,0}(r, \phi, \lambda) = \frac{1}{2} \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{2,0} (3 \sin^2 \phi - 1)$$

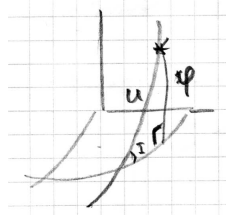


Figure 8.3

2.

$$\frac{\sin U}{\sin 90^\circ} = \frac{\sin \phi}{\sin I}$$

$$\implies \sin \phi = \sin I \sin U$$

$$R_{2,0}(U, I, r) = \frac{1}{2} \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{2,0} (3 \sin^2 I \sin^2 U - 1)$$

3.

$$P_{2,0}(\sin \phi) = \frac{1}{2} (3 \sin^2 \phi - 1) = \frac{1}{2} \left(3 \frac{z^2}{r^2} - 1\right) = \frac{1}{2r^2} (2z^2 - x^2 - y^2)$$

$$\implies R_{2,0}(x, y, z) = \frac{1}{2} \frac{GM}{R} \frac{R^3}{r^5} C_{2,0} (2z^2 - x^2 - y^2)$$

## 8 Perturbation theory

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- Abplattung  $J_2 = -C_{20}$

$$C_{2,0} = -1.08263 \cdot 10^{-3}$$

-

$$\left. \begin{array}{l} \ddot{\mathbf{r}} = \nabla V \\ V = \frac{GM}{r} + R_{2,0} \end{array} \right\} \ddot{\mathbf{r}} = -\frac{GM}{r^3} + \nabla R_{2,0}$$

Q1: Was ist  $R_{2,0}$ ?

Q2: In welchen Koordinatensystemen bildet man den Gradienten?

- Koordinatensysteme  $l$  : lokales Horizontalsystem (links)

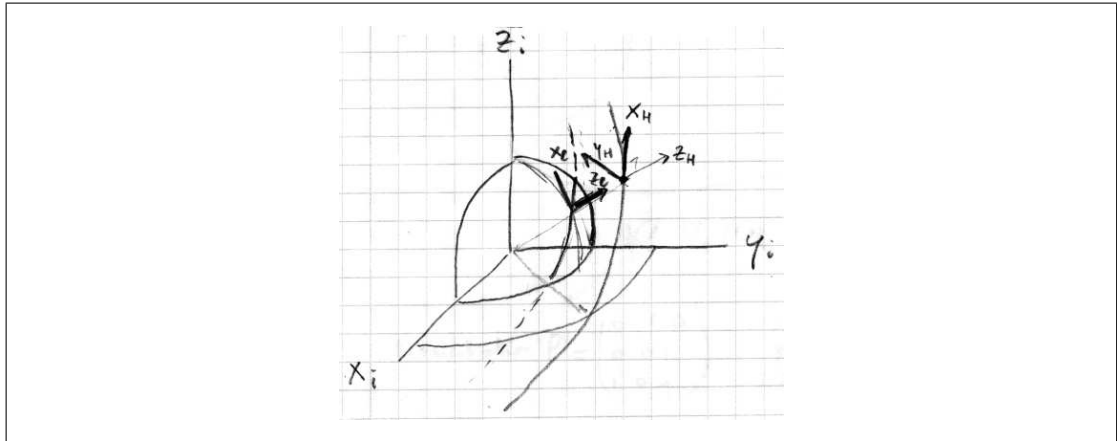


Figure 8.4

$x_l$  - Nord

$y_l$  - Ost

$z_l$  - radial

$H$  : lokales Hill-System (Bahn/mitbewegt)

$x_H$  - komplementaer /  $\sim$  Flugrichtung

$y_H$  -  $L$ -Richtung

$z_H$  - radial

$i$  : inertial geozentrisch

$x_i$  - Fruehlingspunkt  $\Upsilon$

$y_i$  - komplementaer

$z_i$  - konventioneller Himmelspol

$$\nabla_i = \begin{pmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial z_i} \end{pmatrix}, \quad \nabla_l = \begin{pmatrix} \frac{\partial}{r \frac{\partial \phi}{\partial \lambda}} \\ \frac{\partial}{\partial r} \end{pmatrix}, \quad \nabla_H \approx \begin{pmatrix} \frac{\partial}{r \frac{\partial u}{\partial \phi}} \\ \frac{\partial}{r \cos \phi \frac{\partial l}{\partial r}} \end{pmatrix}$$

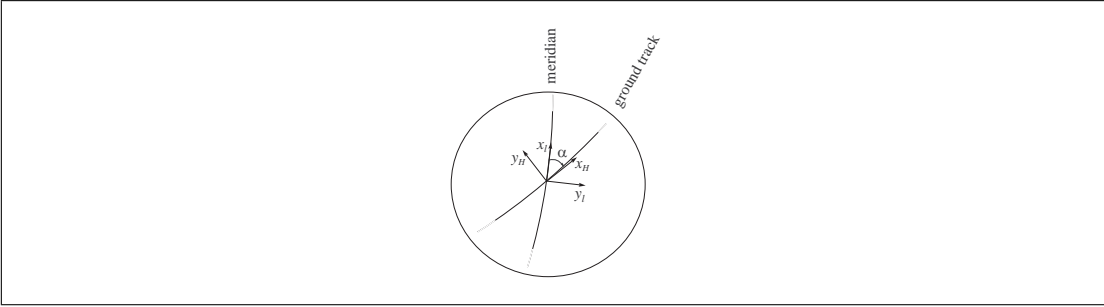


Figure 8.5

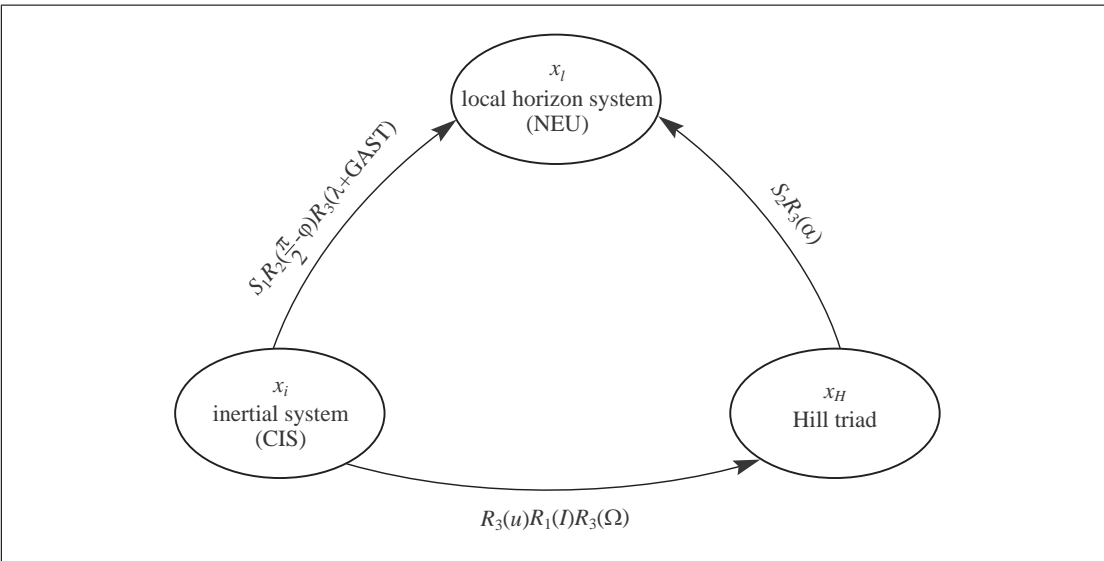


Figure 8.6

i

$$R_{2,0}(r, \phi) = \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{2,0} P_{2,0}(\sin \phi)$$

$$= \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{2,0} \frac{1}{2} (3 \sin^2 \phi - 1)$$

ii

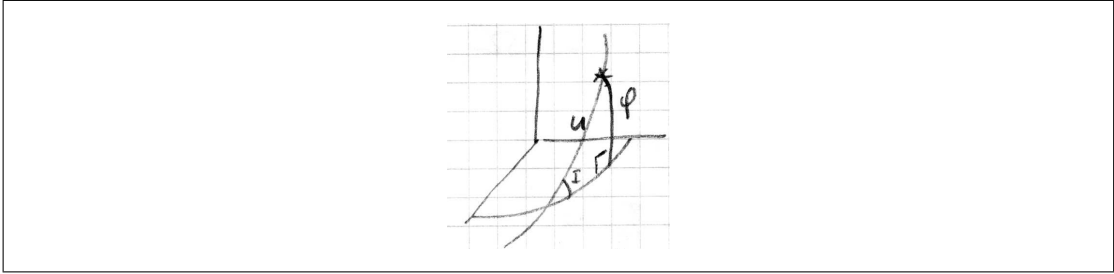


Figure 8.7

$$\frac{\sin u}{\sin 90^\circ} = \frac{\sin \phi}{\sin I}$$

$$\implies \sin \phi = \sin u \sin I$$

$$R_{2,0}(r, u, I) = \frac{1}{2} \frac{GM}{r} \left(\frac{R}{r}\right)^3 C_{2,0} (3 \sin^2 I \sin^2 u - 1)$$

iii

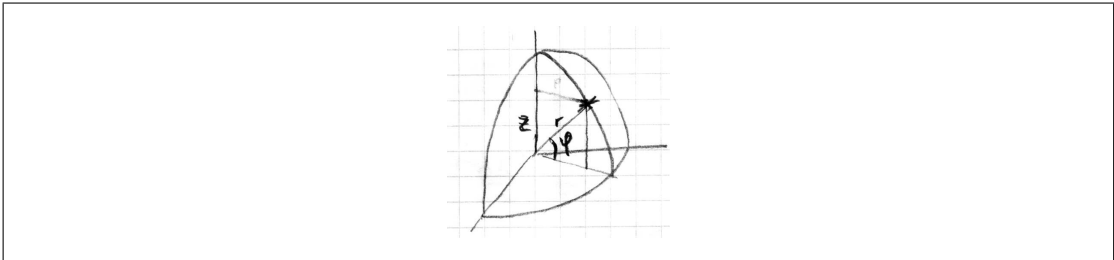


Figure 8.8

$$\sin \phi \frac{z}{r}$$

$$R_{2,0}(x, y, z) = \frac{1}{2} \frac{GM}{R} \left(\frac{R}{r}\right)^3 C_{2,0} (3 \frac{z^2}{r^2} - 1)$$

$$\begin{aligned}
&= \frac{1}{2} \frac{GM}{R} \frac{R^3}{r^5} C_{2,0} (3z^2 - r^2) \\
&= \frac{1}{2} \frac{GM}{R} \frac{R^3}{r^5} C_{2,0} (2z^2 - x^2 - y^2) \\
R_{2,0}(x, y, z) &= Kr^{-5} (2z^2 - x^2 - y^2)
\end{aligned}$$

zusammen:

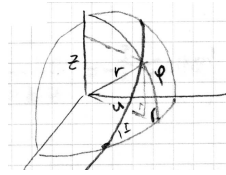


Figure 8.9

$$\frac{1}{r^5} (2z^2 - x^2 - y^2) = f \cdot g$$

$$\begin{aligned}
\frac{\partial R}{\partial x} &= \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \\
&= \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} g + f \frac{\partial g}{\partial x} \\
&= -S \frac{1}{r^6} \frac{x}{r} (2z^2 - x^2 - y^2) + \frac{1}{r^5} (-2x) \\
&= \frac{x}{r^5} \left( \frac{-5}{r^2} (3z^2 - r^2) - 2 \right) \\
&= \frac{x}{r^5} \left( -5 \left( \frac{3z^2}{r^2} - 1 \right) - 2 \right) \\
&= \frac{x}{r^5} \left( -5 \frac{3z^2}{r^2} + 5 - 2 \right) \\
&= \frac{x}{r^5} \left( -5 \frac{z^2}{r^2} + 3 \right) \\
\frac{\partial R_{2,0}}{\partial z} &= -\frac{3}{2} GM \left( \frac{R}{r} \right)^2 C_{2,0} \frac{x}{r^3} \left( 5 \frac{z^2}{r^2} - 1 \right)
\end{aligned}$$

# 9 Non gravitational orbit perturbations

## 9.1 Solar radiation pressure

Solarfluss (Photonenfluss)

$$\Phi = \frac{\Delta E}{A\Delta t} = \text{Menge der Energie, die in } \Delta t \text{ durch Flaeche } A \text{ fliesst}$$

Impuls (linear momentum) eines Photons

$$p_\nu = \frac{E_\nu}{c} \quad (E_\nu = m_\nu c^2 = m_\nu c \cdot c = p_\nu c)$$

$$\implies \Delta p = \frac{\Delta E}{c} = \frac{\Phi}{c} A \Delta t \quad \text{linear momentum}$$

$$\implies F = \frac{\Delta p}{\Delta t} \left( = \frac{dp}{dt} \right) = \frac{\Phi}{c} A \quad \text{Kraft}$$

$$\implies P = \frac{F}{A} = \frac{\Phi}{c} \quad \text{Druck}$$

Fuer erdnahe Satelliten:  $\Phi \approx 1367 \frac{\text{W}}{\text{m}^2}$   
(1 AU)  $\implies P_\odot \approx 4.56 \cdot 10^{-6} \frac{\text{N}}{\text{m}^2}$   $\odot = \text{sun}$

### Orientierung der Flaeche, Absorption oder Reflektion

Absorption: Photonen werden absorbiert

$$\mathbf{F}_{\text{abs}} = -P_\odot \cos \theta A \mathbf{e}_\odot$$

Reflektion

$$\mathbf{F}_{\text{refl}} = -2P_\odot \cos \theta A \cos \theta \mathbf{n}$$

Teils  $(1 - \epsilon)$  Absorption, teils  $(\epsilon)$  reflektiert

$$\implies \mathbf{F}_{\text{a+r}} = -P_\odot \cos \theta A [(1 - \epsilon) \mathbf{e}_\odot + 2\epsilon \cos \theta \mathbf{n}]$$



**Abstand zur Sonne variiert**

z. B.

$$e_{\text{Erde}} \implies \left. \begin{aligned} a(1-e) &= 147Gm \\ a(1+e) &= 152Gm \end{aligned} \right\} \Phi \text{ variiert } \pm 3.3\%$$

$$\mathbf{f} = \frac{F}{m} = -P_{\odot} \left( \frac{1\text{AU}}{r_{\odot}} \right)^2 \frac{A}{m} \cos \theta [(1-\epsilon)\mathbf{e}_{\odot} + 2\epsilon \cos \theta \mathbf{n}]$$

Vereinfacht, z. B. grosse Sonnensegel  $\perp$  Sonne:

$$\mathbf{n} = \mathbf{e}_{\odot} = \frac{\mathbf{r}_{\odot}}{r_{\odot}}$$

$$\implies \mathbf{f} = -P_{\odot} C_R \frac{a}{m} \frac{\mathbf{r}_{\odot}}{r_{\odot}^3} \text{AU}^2$$

$$C_R = 1 + \epsilon$$

Beispiel:

| Material                        | $\epsilon$ (Refl.) | $1 - \epsilon$ (Abs.) | $C_R = 1 + \epsilon$ |
|---------------------------------|--------------------|-----------------------|----------------------|
| Sonnensegel                     | 0.21               | 0.79                  | 1.21                 |
| Antenne                         | 0.30               | 0.70                  | 1.30                 |
| mit Al beschichtete Sonnensegel | 0.88               | 0.12                  | 1.88                 |

**Schattenfunktion  $\chi$**

$$\mathbf{f} = \chi P_{\odot} \left( \frac{1\text{AU}}{r_{\odot}} \right)^2 \frac{A}{m} \cos \theta [(1-\epsilon)\mathbf{e}_{\odot} + 2\epsilon \cos \theta \mathbf{n}]$$

**9.2 Atmospheric drag**

haengt ab von

- Satellitengeometrie
- Satellitengeschwindigkeit
- Atmosphaerengeschwindigkeit
- atmosphaerische Dichte, Temperatur, Zusammensetzung
- Aerodynamik

⇒ empirische Modellierung

$$\Delta m = \rho \Delta V = \rho A v \Delta t$$

Annahme:

$$\Delta p \sim -\Delta m V \sim -\rho A v^2 \Delta t$$

$$\Rightarrow F = \frac{\Delta p}{\Delta t} \sim -\rho A v^2$$

$$F = m f \Rightarrow f \sim -\rho \frac{A}{m} V^2$$

$$\mathbf{f}_{\text{drag}} = -\frac{1}{2} C_D \rho(\mathbf{r}, t) \frac{A}{m} (\mathbf{V} - \mathbf{V}_{\text{atm}}) |\mathbf{V} - \mathbf{V}_{\text{atm}}|$$

- $C_D$  ← hat mit Form zu tun (Windkanal!)  
 Kugel:  $C_D = 1$   
 Praxis: 0
- $\mathbf{V}_{\text{atm}} \approx \rho_0 e^{-\frac{h}{H_0}}$  ← sehr grob  
 ⇒ Tabelle 33 in Seeber
- $C_D \frac{A}{m}$  = "ballistischer Koeffizient"